Trig Identities

Definition: An identity is a statement that is true for all values of the domain of the variable.

eg. $x + 8 = 8 + x$

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

*Note: We write $\sin^2\theta$ instead of $(\sin\theta)^2$.

Proving Identities

Solving equations is different than proving an identity. When we prove an identity, we are trying to work with one side to transform it into the other side.

Example: Prove the identity.

$$\frac{\sin^2 x - 1}{\sin^2 x} = -\cot^2 x$$

Solution:

Decide which side to work with. Then start with an equation that sets that side equal to itself. Then work with one of the sides until you attain what you are trying to prove.

$$\begin{aligned}\frac{\sin^2 x - 1}{\sin^2 x} &= \frac{\sin^2 x - 1}{\sin^2 x} \\ &= \frac{(1 - \cos^2 x) - 1}{\sin^2 x} \\ &= \frac{-\cos^2 x}{\sin^2 x} \\ &= -\cot^2 x\end{aligned}$$

Alternate Solution:

$$\frac{\sin^2 x - 1}{\sin^2 x} = \frac{\sin^2 x - 1}{\sin^2 x}$$

$$= \frac{\sin^2 x}{\sin^2 x} - \frac{1}{\sin^2 x}$$

$$= 1 - \csc^2 x$$

$$= 1 - (1 + \cot^2 x)$$

$$= -\cot^2 x$$

Note: There is often more than one way to prove a trigonometric identity.

Example: Prove the identity.

$$\frac{1}{\sec x - 1} - \frac{1}{\sec x + 1} = 2\cot^2 x$$

We usually start with the more complicated side, so we will work with the left. Start by getting a common denominator and adding the fractions.

$$\frac{1}{\sec x - 1} - \frac{1}{\sec x + 1} = \frac{1}{\sec x - 1} - \frac{1}{\sec x + 1}$$

$$= \frac{1}{(\sec x - 1)} \cdot \frac{(\sec x + 1)}{(\sec x + 1)} - \frac{1}{(\sec x + 1)} \cdot \frac{(\sec x - 1)}{(\sec x - 1)}$$

$$= \frac{(\sec x + 1) - (\sec x - 1)}{(\sec x - 1)(\sec x + 1)}$$

$$= \frac{(\sec x + 1) - (\sec x - 1)}{(\sec^2 x - 1)}$$

$$= \frac{2}{\tan^2 x}$$

$$= 2 \cdot \frac{1}{\tan^2 x}$$

$$= 2\cot^2 x$$

Example: Prove the identity.

$$(1+\cot^2 x)(1-\sin^2 x) = \cot^2 x$$

Work with the left side. Notice that you have some Pythagorean identities.

$$\begin{aligned}(1+\cot^2 x)(1-\sin^2 x) &= (1+\cot^2 x)(1-\sin^2 x)\\ &= \csc^2 x \cos^2 x\\ &= \left(\frac{1}{\sin^2 x}\right)\cos^2 x\\ &= \frac{\cos^2 x}{\sin^2 x}\\ &= \cot^2 x\end{aligned}$$

Example: Prove the identity.

$$\sec u + \tan u = \frac{1}{\sec u - \tan u}$$

If you don't know what else to do, convert all terms to sine and cosine. Work with the left side.

$$\sec u + \tan u = \sec u + \tan u$$

$$= \frac{1}{\cos u} + \frac{\sin u}{\cos u}$$

$$= \frac{1 + \sin u}{\cos u}$$

Notice that the numerator is close to the Pythagorean Identity. See if you can make it look like it.

$$= \frac{1 + \sin u}{\cos u} \cdot \frac{(1 - \sin u)}{(1 - \sin u)}$$

$$= \frac{1 - \sin^2 u}{\cos u(1 - \sin u)}$$

$$= \frac{\cos^2 u}{\cos u(1 - \sin u)}$$

$$= \frac{\cos u}{(1 - \sin u)}$$

Now look at what you are trying to prove. You need a 1 in the numerator and now have cos u. To make cos u = 1, you need to multiply it by 1/cos u. This is the same as sec u.

$$= \frac{\cos u}{(1 - \sin u)} \cdot \frac{(\sec u)}{(\sec u)}$$

$$= \frac{1}{\sec u - \sin u \sec u}$$

$$= \frac{1}{\sec u - \sin u \dfrac{1}{\cos u}}$$

$$= \frac{1}{\sec u - \dfrac{\sin u}{\cos u}}$$

$$= \frac{1}{\sec u - \tan u}$$

Sometimes you need to look at what you are trying to get to decide on the next step.

Example: Prove the identity.

$$\cot t \cos t = \csc t - \sin t$$

Take the right side and try writing it in terms of sine and cosine.

$$\begin{aligned}
\csc t - \sin t &= \csc t - \sin t \\
&= \frac{1}{\sin t} - \sin t \\
&= \frac{1}{\sin t} - \sin t \cdot \frac{\sin t}{\sin t} \\
&= \frac{1 - \sin^2 t}{\sin t} \\
&= \frac{\cos^2 t}{\sin t} \\
&= \frac{\cos t}{\sin t} \cdot \cos t \\
&= \cot t \cos t
\end{aligned}$$

Example: Prove the identity.

$$\frac{\cot^2 u}{1+\csc u} = \frac{1-\sin u}{\sin u}$$

Start with the left side.

$$\begin{aligned}\frac{\cot^2 u}{1+\csc u} &= \frac{\cot^2 u}{1+\csc u} \\ &= \frac{\csc^2 u - 1}{1+\csc u} \\ &= \frac{(\csc u + 1)(\csc u - 1)}{1+\csc u} \\ &= \csc u - 1\end{aligned}$$

We seem to be stuck. Start over and work with the right side.

$$\begin{aligned}\frac{1-\sin u}{\sin u} &= \frac{1-\sin u}{\sin u} \\ &= \frac{1}{\sin u} - \frac{\sin u}{\sin u} \\ &= \csc u - 1\end{aligned}$$

Notice that we got to the same place with both sides.

Start over and put the 2 parts together.

$$\frac{\cot^2 u}{1+\csc u} = \frac{\cot^2 u}{1+\csc u}$$
$$= \frac{\csc^2 u - 1}{1+\csc u}$$
$$= \frac{(\csc u + 1)(\csc u - 1)}{1+\csc u}$$
$$= \csc u - 1$$
$$= \frac{1}{\sin u} - \frac{\sin u}{\sin u}$$
$$= \frac{1-\sin u}{\sin u}$$

Sometimes it is helpful to work with both sides until you get something that matches. Then write the verification working with only one side, but using the information you gained from working the other side, too.

There is no well-defined set of rules to follow in verifying trig identities. Practice is your best weapon. Here are some guidelines that are helpful:

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.

2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.

3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.

4. If the preceding guidelines do not help, try converting all terms to sines and cosines.

5. Always try something. Even paths that lead to dead ends give you insights.

Sum and Difference Formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$
$$\cos 2u = \cos^2 u - \sin^2 u$$
$$\sin 2u = 2 \sin u \cos u$$

*See derivations on p. 468-69.

These formulas help us to find exact values of trig functions involving sums and differences of special angles from our unit circle.

Example: Find the exact value of sin 15°.

$$\sin 15° = \sin(45° - 30°)$$

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example: Find the exact value of $\cos\frac{7\pi}{12}$.

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Example: Find the exact value of cos 105°.

$$\begin{aligned}\cos 105° &= \cos(45°+60°)\\ &= \cos 45°\cos 60°-\sin 45°\sin 60°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Example: Find the exact value of the following:

$$\cos 78° \cos 18° + \sin 78° \sin 18°$$

This looks like it came from

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

So we have

$$\cos 78° \cos 18° + \sin 78° \sin 18° = \cos(78° - 18°) = \cos 60° = \frac{1}{2}$$

Example: Find the exact value of

sin10°cos35°+cos 10°sin35°

This looks like it came from

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

So we have

$$\sin 10° \cos 35° + \sin 10° \sin 35° = \sin(10° + 35°) = \sin 45° = \frac{\sqrt{2}}{2}$$

Example: Verify the cofunction identity sin(90°- x)=cos x.

$$\begin{aligned}\sin(90° - x) &= \sin 90° \cos x - \cos 90° \sin x \\ &= 1 \cdot \cos x - 0 \cdot \sin x \\ &= \cos x\end{aligned}$$

Example: Prove the identity:

$$\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$$

$\sin(x + y)\sin(x - y)$

$= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$

**This is a difference of squares.*

$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$

**Substitute for the $\sin^2 x$ and $\sin^2 y$.*

$= (1 - \cos^2 x)\cos^2 y - \cos^2 x(1 - \cos^2 y)$

**Multiply using Distributive.*

$= \cos^2 y - \cos^2 x \cos^2 y - \cos^2 x + \cos^2 x \cos^2 y$

**Combine like terms.*

$= \cos^2 y - \cos^2 x$

Example: Prove the identity.

$$2\cot 2x + \tan x = \cot x$$

<u>Proof</u>:

$$\begin{aligned}
2\cot 2x + \tan x &= 2\cot 2x + \tan x \\
&= \frac{2\cos 2x}{\sin 2x} + \frac{\sin x}{\cos x} \\
&= \frac{2(\cos^2 x - \sin^2 x)}{2\sin x\cos x} + \frac{\sin x}{\cos x} \\
&= \frac{2(\cos^2 x - \sin^2 x)}{2\sin x\cos x} + \frac{\sin x}{\cos x} \cdot \frac{(2\sin x)}{(2\sin x)} \\
&= \frac{2\cos^2 x - 2\sin^2 x + 2\sin^2 x}{2\sin x\cos x} \\
&= \frac{2\cos^2 x}{2\sin x\cos x} \\
&= \frac{\cos x}{\sin x} \\
&= \cot x
\end{aligned}$$

Example: Prove the identity.

$$\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$$

Proof:

$$\frac{\sin x}{1-\cos x} = \frac{\sin x}{1-\cos x}$$

Note: If we can get 1- cos^{2}x on the bottom, it can be written as sin^{2}x. Use the conjugate.

$$\begin{aligned}\frac{\sin x}{1-\cos x} &= \frac{\sin x}{1-\cos x} \\ &= \frac{\sin x}{1-\cos x} \cdot \frac{(1+\cos x)}{(1+\cos x)} \\ &= \frac{\sin x(1+\cos x)}{1-\cos^2 x} \\ &= \frac{\sin x(1+\cos x)}{\sin^2 x} \\ &= \frac{(1+\cos x)}{\sin x}\end{aligned}$$