Inverse Functions

Look at the function f = {(1,5), (2,6), (3, 7), (4, 8)}

If we interchange the coordinates we get:

{(5, 1), (6, 2), (7, 3), (8, 4)}

This is called the inverse function of the function f and we use the notation f^{-1}.

So, f^{-1} = {(5, 1), (6, 2), (7, 3), (8, 4)}

Sample Problems:

Find the following:

a) f(1) answer: 5

b) $f^{-1}(6)$ answer: 2

c) $f(f^{-1}(6))$ answer: 6

Example: Find the inverse of $y = 2x - 3$.

1. Exchange the x and y: $x = 2y - 3$

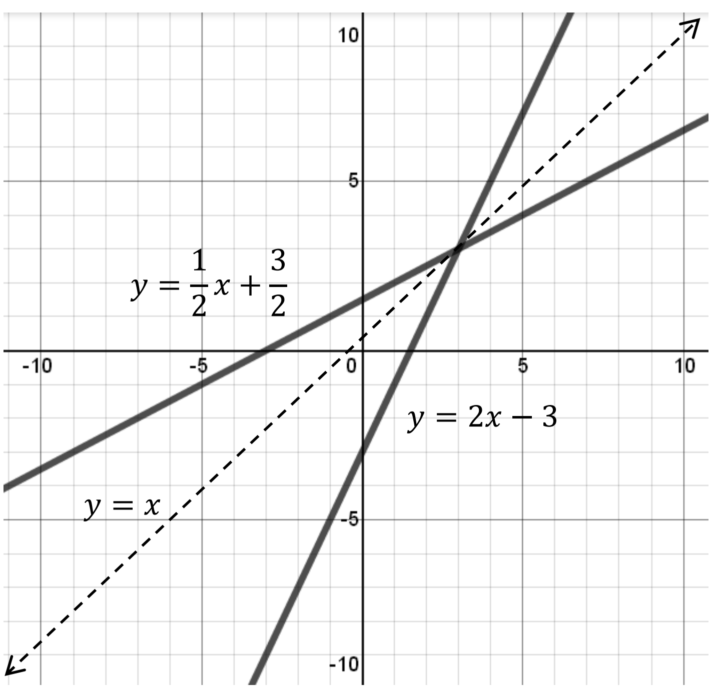
2. Solve for y.

$$2y = x + 3$$

$$\frac{2y}{2} = \frac{x}{2} + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{3}{2}$$

Look at the graphs:



The inverse of a function is its reflection over the line $y = x$.

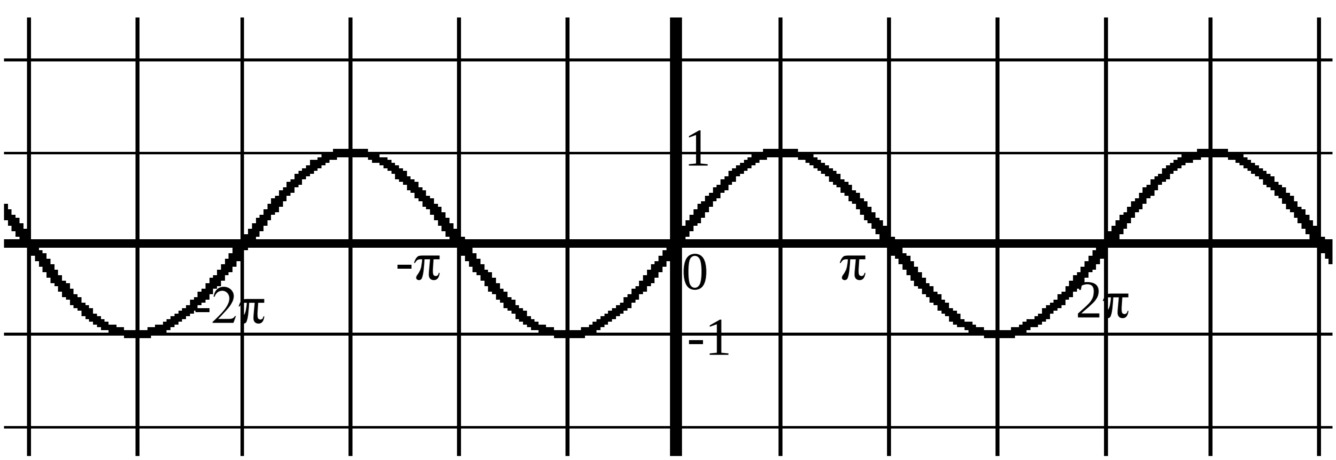
Inverse Trig Functions

Remember the table for $y = \sin x$.

x	y
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0

x	y
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2} \approx -0.7$
$\frac{4\pi}{3}$	$\frac{-\sqrt{3}}{2} \approx -0.87$
$\frac{3\pi}{2}$	-1
$\frac{2\pi}{3}$	$\frac{-\sqrt{3}}{2} \approx -0.87$
$\frac{7\pi}{4}$	$\frac{-\sqrt{2}}{2} \approx -0.7$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

x	y
0	0
$\frac{13\pi}{6}$	$\frac{1}{2}$
$\frac{9\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
$\frac{7\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{5\pi}{2}$	1
$\frac{8\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{11\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
$\frac{17\pi}{6}$	$\frac{1}{2}$
3π	0



Now, let's look at the inverse of the sine function. To get this, we have to reverse the x and y values in the table:

x	y
0	0
$\frac{1}{2}$	$\frac{\pi}{6}$
$\frac{\sqrt{2}}{2} \approx 0.7$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{\pi}{3}$
1	$\frac{\pi}{2}$
$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{2\pi}{3}$
$\frac{\sqrt{2}}{2} \approx 0.7$	$\frac{3\pi}{4}$
$\frac{1}{2}$	$\frac{5\pi}{6}$
0	π

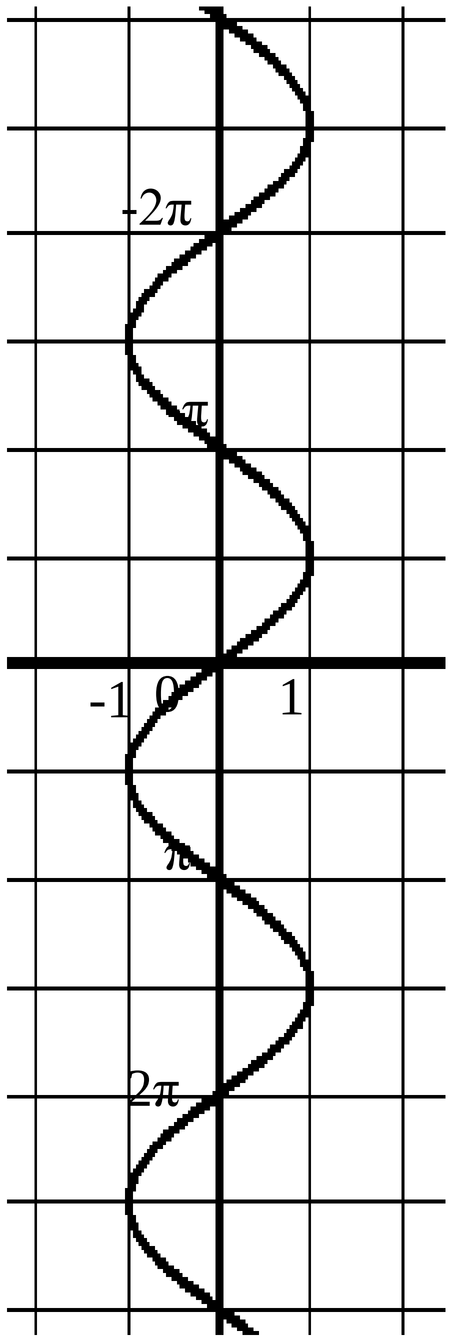
x	y
0	π
-$\frac{1}{2}$	$\frac{7\pi}{6}$
$\frac{-\sqrt{2}}{2} \approx -0.7$	$\frac{5\pi}{4}$
$\frac{-\sqrt{3}}{2} \approx -0.87$	$\frac{4\pi}{3}$
-1	$\frac{3\pi}{2}$
$\frac{-\sqrt{3}}{2} \approx -0.87$	$\frac{2\pi}{3}$
$\frac{-\sqrt{2}}{2} \approx -0.7$	$\frac{7\pi}{4}$
-$\frac{1}{2}$	$\frac{11\pi}{6}$
0	2π

x	y
0	0
$\frac{1}{2}$	$\frac{13\pi}{6}$
$\frac{\sqrt{2}}{2} \approx 0.7$	$\frac{9\pi}{4}$
$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{7\pi}{3}$
1	$\frac{5\pi}{2}$
$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{8\pi}{3}$
$\frac{\sqrt{2}}{2} \approx 0.7$	$\frac{11\pi}{4}$
$\frac{1}{2}$	$\frac{17\pi}{6}$
0	3π

The inverse of $y = \sin x$ is $x = \sin y$. Since we can't solve this for y like we usually do for inverses, we need a new notation:

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x$$

Here's the graph:



Notice that this is not a function. In order for this to be a function, we just consider part of it.

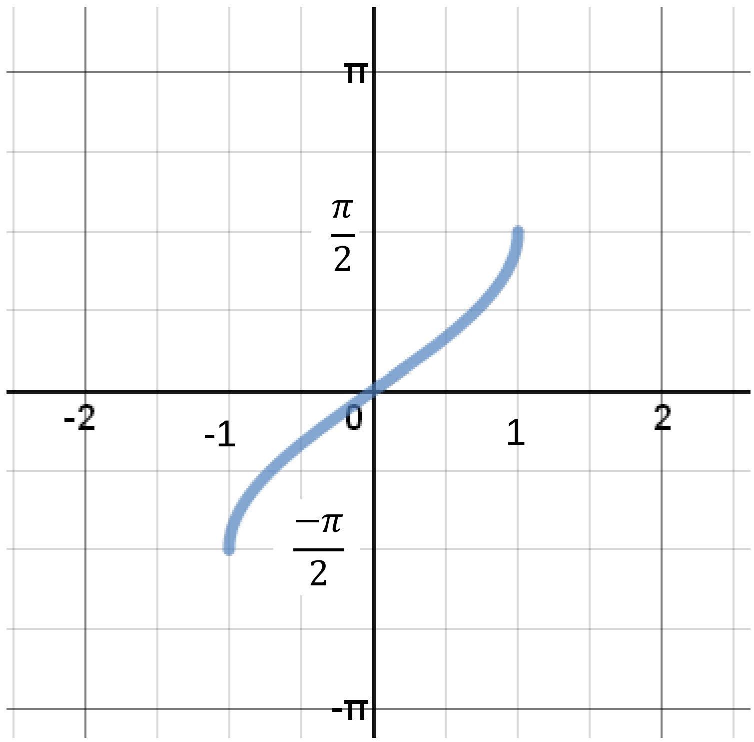
Definition: The <u>inverse sine function</u> can be denoted by

$y = \arcsin x$ if and only if $x = \sin y$

where -1 ≤ x ≤ 1 and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$.

It can be thought of as the angle whose sine is x.

Note: This gives us the following graph:

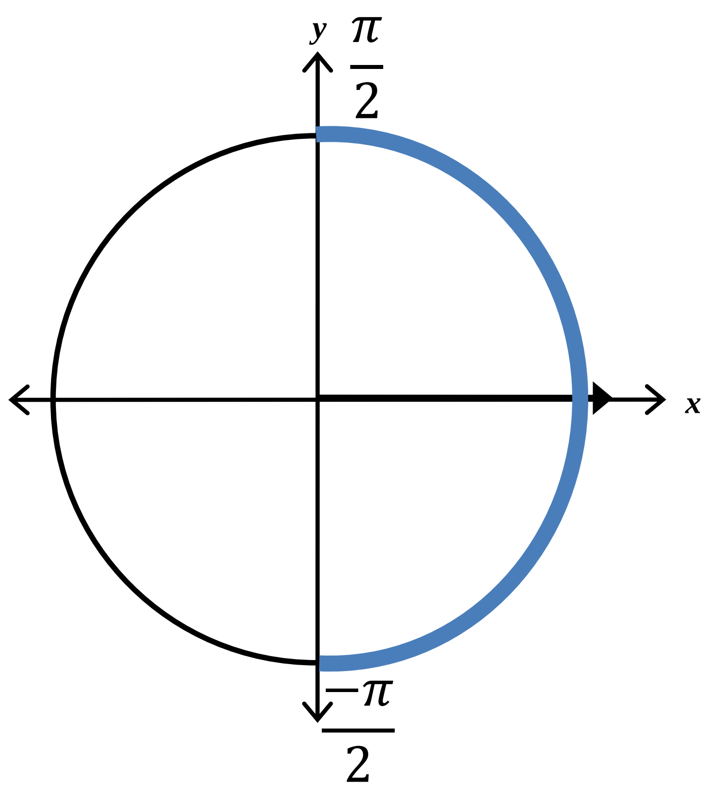


In your textbook, if we want all values y for any given x, we write

$$y = \arcsin x$$

If we want only the principal values, we restrict the range to $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ and we write

$$y = \text{Arcsin}\, x \text{ or } y = \text{Sin}^{-1} x$$



The "zone" for $y = \text{Arcsin}\, x$

Example:

$$y = \arcsin\frac{1}{2}$$

Ask "Where is the sine equal to $\frac{1}{2}$?"

Answer: On the unit circle, the sine equals $\frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ and all coterminal angles. So, to include them we write:

$$x = \frac{\pi}{6} \pm 2n\pi \text{ or } x = \frac{5\pi}{6} \pm 2n\pi$$

Example:

$$y = \text{Arcsin}\frac{1}{2}$$

Ask "Where is the sine equal to $\frac{1}{2}$?"

Answer: On the unit circle, the sine equals $\frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Because it is $\underline{\text{Arcsin}}$, we only want the answer in the "zone."

$$x = \frac{\pi}{6}$$

Find the following:

a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ answer: $x = \frac{\pi}{3} \pm 2n\pi$ or
$x = \frac{2\pi}{3} \pm 2n\pi$

b) $\arcsin 1$ answer: $x = \frac{\pi}{2} \pm 2n\pi$

c) $y = \text{Arcsin}\frac{-\sqrt{2}}{2}$ answer: $\frac{-\pi}{4}$ (not $\frac{7\pi}{4}$)

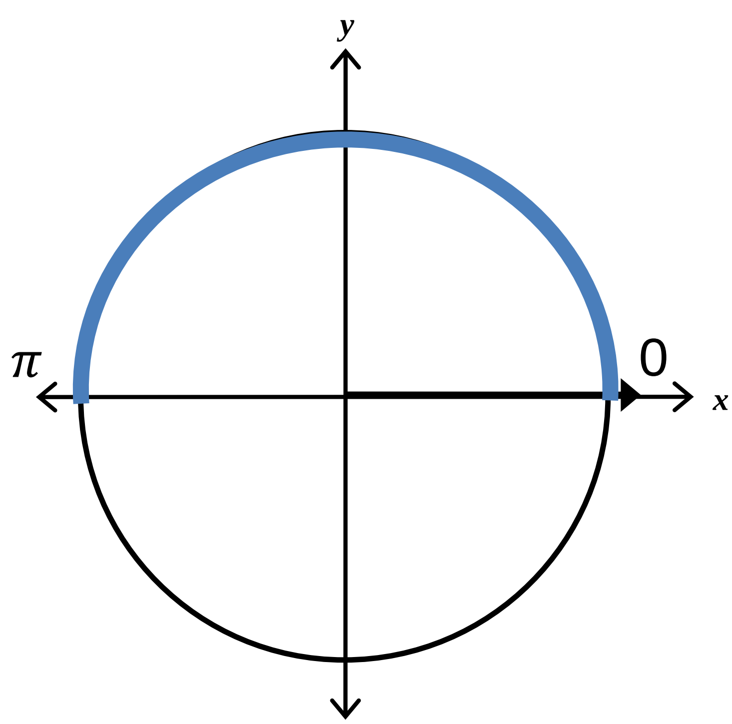
d) $\text{Sin}^{-1}(-1)$ answer: $\frac{-\pi}{2}$ (not $\frac{3\pi}{2}$)

The inverse functions for the other trig functions are similar.

Function	*Domain*	*Range*
$y = \text{Arcsin } x \leftrightarrow \sin y = x$	[-1, 1]	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
$y = \text{Arccos } x \leftrightarrow \cos y = x$	[-1, 1]	$[0, \pi]$
$y = \text{Arctan x} \leftrightarrow \tan y = x$	[-∞,∞]	$(\frac{-\pi}{2}, \frac{\pi}{2})$

Notice that the "zone" for Arcsin and Arctan are the same.

Arccos has a different "zone."



The "zone" for $y = \text{Arccos}\, x$

Example: Evaluate the following.

a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

answer: $x = \frac{\pi}{6} \pm 2n\pi$ or

$x = \frac{11\pi}{6} \pm 2n\pi$

Note: You could also write this as $x = \frac{\pm\pi}{6} \pm 2n\pi$

b) Arctan (-1)

answer: $\frac{-\pi}{4}$ (not $\frac{3\pi}{4}$)

c) $\arccos\left(\frac{-1}{2}\right)$

answer: $\frac{2\pi}{3}$

d) Tan^{-1} 0

answer: $x = 0 \pm 2n\pi$

(or just $\pm 2n\pi$)

e) arctan 1

answer: $x = \frac{\pi}{4} \pm n\pi$

Remember: tangent has a period of π, so you only add multiples of π to cover all of the answers.

Example: Use a calculator to evaluate the following in degrees. (Set your calculator mode to radians.)

a) Sin^{-1} 0.5587 answer: [2nd] [sin] 0.5587 [ENTER]
≈ 34°

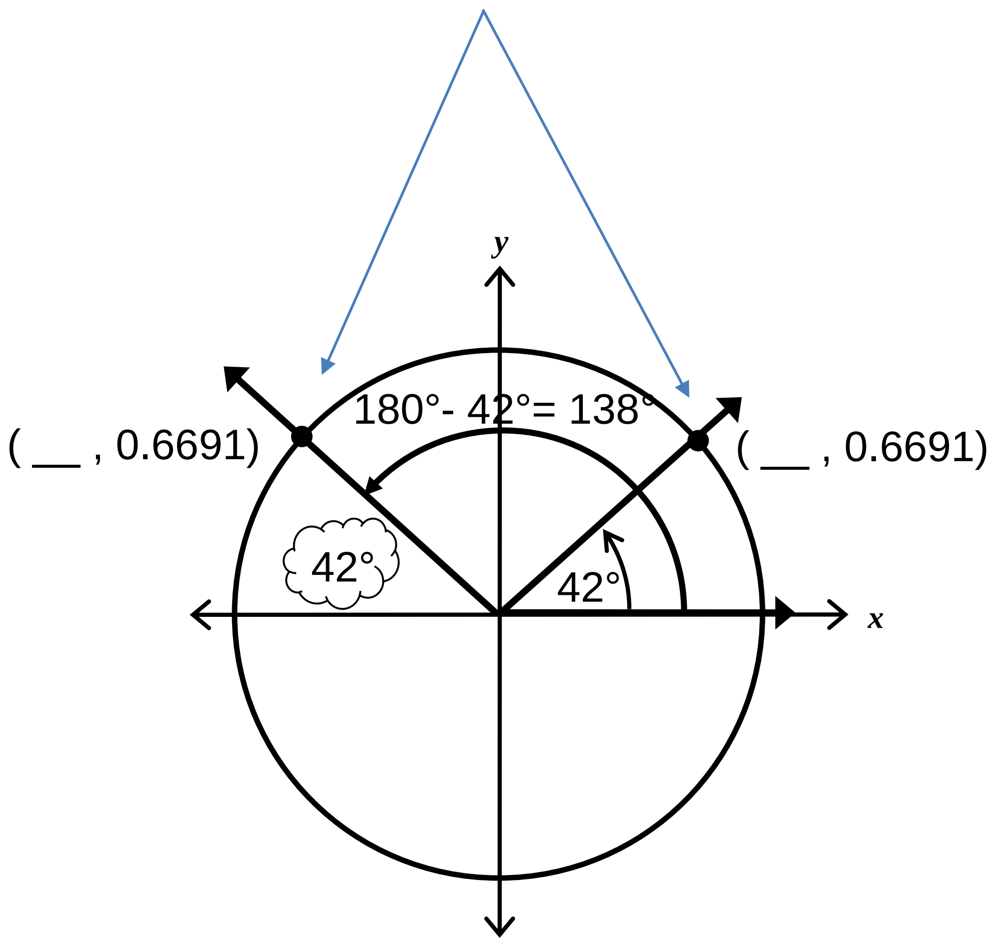
b) Tan^{-1} -3.254 answer: [2nd] [tan] -3.254 [ENTER]
≈ -73°

c) Arcos 0.2345 answer: [2nd] [cos] 0.2345 [ENTER]
≈ 76°

d) sin 0.6691 answer: [2nd] [sin] 0.6691 [ENTER]
≈ 42°

Note: since we need to include <u>all</u> angles with this sine, we need to look at our circle and find the other angle with the same sign.

These 2 points have the same y-coordinates, which means they both have the same sine.



Use the reference angle to find the other angle's measure.

180°- 42°= 138°

Our answer, then, is $x = 42° \pm 360n$ or $x = 138° \pm 360n$

Example:

Look at $\cos x = \frac{\sqrt{2}}{2}$

Since $\cos^{-1}$ and cos are inverse functions, they undo each other. So, for the above equation, we can take $\cos^{-1}$ of both sides to solve the equation.

$$cos^{-1}(\cos x) = cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$x = \frac{\pm\pi}{4} \pm 2n\pi$$