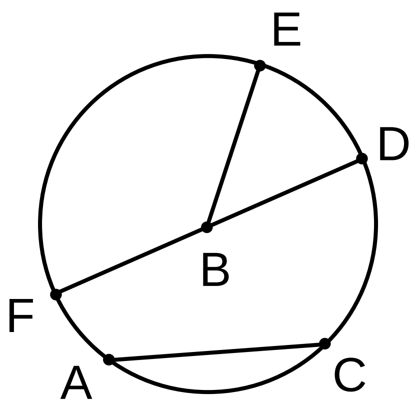
section 9.1

Definitions:

circle – the set of all points that are given distance from a given point in a given plane



Notation: ⊙B

center – the given point in the plane

radius of a circle – a segment that connects a point on the circle with the center (pl. radii)

chord of a circle – a segment having both endpoints on the circle

diameter – a chord that passes through the center of a circle

arc – a curve that is a subset of a circle

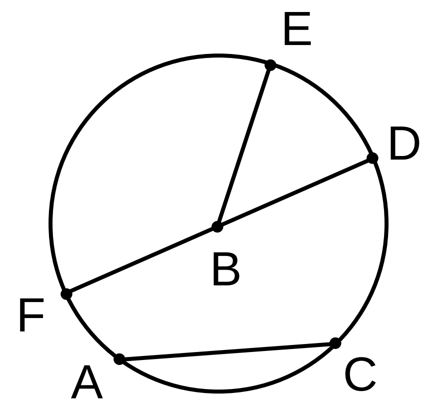
<u>center</u>: B

<u>radii</u>: $\overline{BE}$, $\overline{BD}$, $\overline{BF}$

<u>chords</u>: $\overline{FD}$, $\overline{AC}$

<u>diameter</u>: $\overline{FD}$

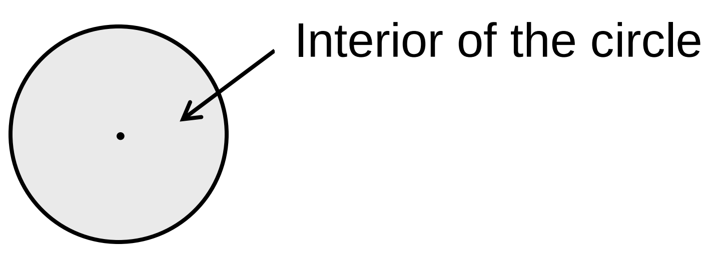
<u>arc</u>: $\overset{\frown}{ED}$, $\overset{\frown}{DC}$, $\overset{\frown}{FC}$, etc.



Definitions:

<u>interior of a circle</u> - the set of all planar points whose distance from the center of the circle is less than the length of the radius

<u>exterior of a circle</u> – the set of all planar points whose distance from the center is greater than the length of the radius



Note: The radius, diameter, and chord are segments associated with circles but they are not part of the circle. they may have endpoints on the circle and thus share a point or two, but are not part of the cir le. The center is alos not part of the circle.

Note: When we refer to the area of a circle, we are referring to the area of the circular region. The circle itself is a curve and has no area.

*By definition of a circle, we know <u>all radii of a circle are congruent</u>.

Definition: <u>Congruent circles</u> are circles whose radii are congruent.

Chord Postulate (9.1): If a line intersects the interior of a circle, then it contains a chord of the circle.

Theorem 9.1: In a circle, if a radius is perpendicular to a chord of a circle, then it bisects the chord.

Proof of Thm. 9.1:

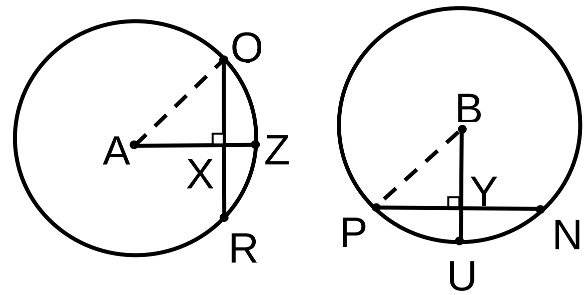
Given: Circle O with radius
$\overline{OC}$ and chord $\overline{AB}$
$\overline{OC} \perp \overline{AB}$
Prove: $\overline{OC}$ bisects $\overline{AB}$

Statement		**Reason**	
1.	Circle O with radius $\overline{OC}$ with chord $\overline{AB}$, $\overline{OC} \perp \overline{AB}$	1.	Given
2.	Draw radii $\overline{OA}$ and $\overline{OB}$	2.	Auxilliary lines
3.	$\angle ODA$ & $\angle ODB$ are rt. angles	3.	Def. of perp. lines
4.	Δ ODA & ΔODB are rt. Δ's	4.	Def. of rt. Δ's
5.	$\overline{OA} \cong \overline{OB}$	5.	Radii of a circle $\cong$
6.	$\overline{OD} \cong \overline{OD}$	6.	Reflexive
7.	$\Delta ODA \cong \Delta ODB$	7.	HL
8.	$\overline{AD} \cong \overline{BD}$	8.	Def. of $\cong$ Δ's
9.	AD = BD	9.	Def of $\cong$ segments
10.	D is midpoint of $\overline{AB}$	10.	Def. of midpoint
11.	$\overline{OC}$ bisects $\overline{AB}$	11.	Def. of seg. bisector
12.	If $\overline{OC} \perp \overline{AB}$, then $\overline{OC}$ bisects $\overline{AB}$	12.	Law of Deduction

<u>Theorem 9.2</u>: In a circle or in congruent circles, if two chords are the same distance from the center(s), the chords are congruent.

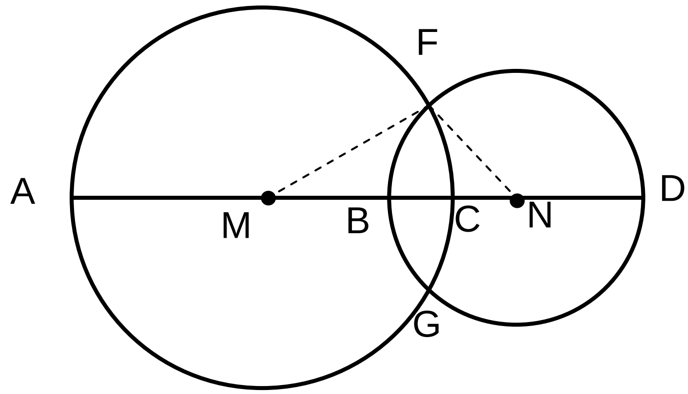
Given: $\odot A \cong \odot B$, $AX = BY$
$\overleftrightarrow{AZ} \perp \overleftrightarrow{QR}$, $\overleftrightarrow{BU} \perp \overleftrightarrow{PN}$

Prove: $\overline{QR} \cong \overline{PN}$



Statement	**Reason**
1. $\odot A \cong \odot B$, $\overleftrightarrow{AZ} \perp \overleftrightarrow{QR}$, $\overleftrightarrow{BU} \perp \overleftrightarrow{PN}$, $AX = BY$	1. Given
2. $\overline{AX} \cong \overline{BY}$	2. Def. of ≅ segments
3. Draw $\overline{AQ}$ and $\overline{BP}$	3. Line Postulate
4. $\overline{AQ} \cong \overline{BP}$	4. Def. of ≅ circles
5. ∠AXQ & ∠BYP are rt ∠'s	5. Def. of perpendicular
6. ΔAXQ & ΔBYP are rt Δ's	6. Def. of rt. Δ
7. ΔAQX ≅ ΔBPY	7. HL
8. $\overline{QX} \cong \overline{PY}$	8. Def. of ≅ Δ's
9. QX = PY	9. def. of ≅ segments
10. $\overleftrightarrow{AZ}$ bisects $\overleftrightarrow{QR}$, $\overleftrightarrow{BU}$ bisects $\overleftrightarrow{PN}$	10. Radius ⊥ a chord bisects it
11. X midpt of $\overline{QR}$, Y is midpt of $\overline{PN}$	11. Def. of seg. bisector
12. QX = ½ QR, PY = ½ PN	12. Midpoint Theorem
13. ½ QR = ½ PN	13. Substitution (10 into 7)
14. QR = PN	14. Mult Prop. of eq.
15. $\overline{QR} \cong \overline{PN}$	15. Def. of ≅ segments
16. If $\odot A \cong \odot B$, $\overleftrightarrow{AZ} \perp \overleftrightarrow{QR}$, $\overleftrightarrow{BU} \perp \overleftrightarrow{PN}$, AX = BY, then QR ≅ PN	16. Law of Deduction

Sample Problem: Given circle M with radius 6 and circle N with radius 4 as shown in the diagram.



1. If BC = 1, find CN.

   BN = 4 (radius), so CN = BN-BC = 4 -1 = 3

2. If BC = 1, then find the perimeter of ΔMNF.

   MF=6, FN=4 (radii) and MN = 6+4-1=9
   Perimeter = 6+4+9 = 19

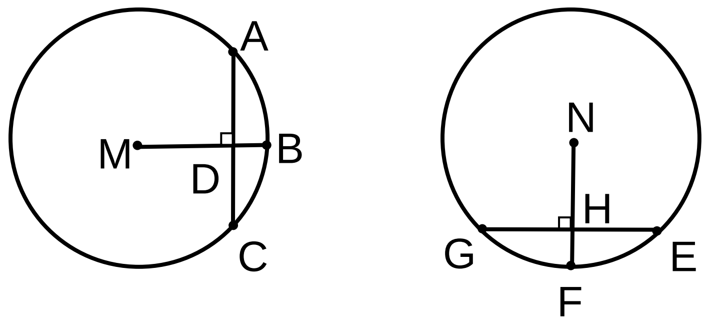
3. If BC = 2, then find AD.

   AD = 12 + 8 – 2 = 18

4. If BC = 2, then find AB.

   AB+BC= 12,
   AB+ 2 = 12
   AB = 10

<u>Sample Problem</u>: Given $\odot M \cong \odot N$, DM = HN, AD = 5x + 4 units, GE = 18 units. Find x.



$\overline{AC}$ and $\overline{GE}$ are bisected by the radius (Thm 9.1) They are also equidistant from their centers, and so must be congruent (Thm 9.2).

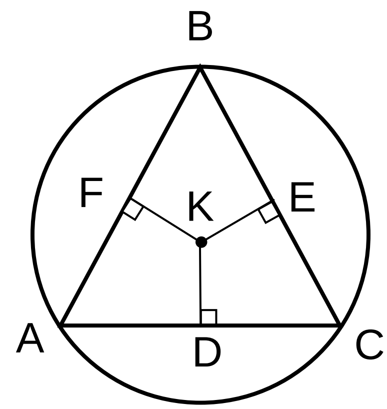
$$AC = GE = 18 \text{ and } AD = \tfrac{1}{2}(18) = 9$$
$$5x + 4 = 9$$
$$5x = 5$$
$$x = 1$$

<u>Theorem 9.3</u>: In a circle or in congruent circles, if two chords are congruent , then they are the same distance from the center(s).

<u>Sample Problem:</u> Prove that if the center of a circle is equidistant from the sides of an inscribed triangle, then the triangle is equilateral.

Given: Center K is equidistant from the sides of inscribed ΔABC

Prove: ΔABC is equilateral

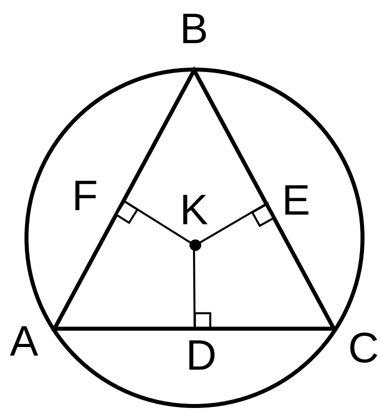


Statement	Reason
1. Center K is equidistant from the sides of inscribed ΔABC	1. Given
2.	2.
3.	3.
4.	4.
5.	5.
6.	6.
7.	7.

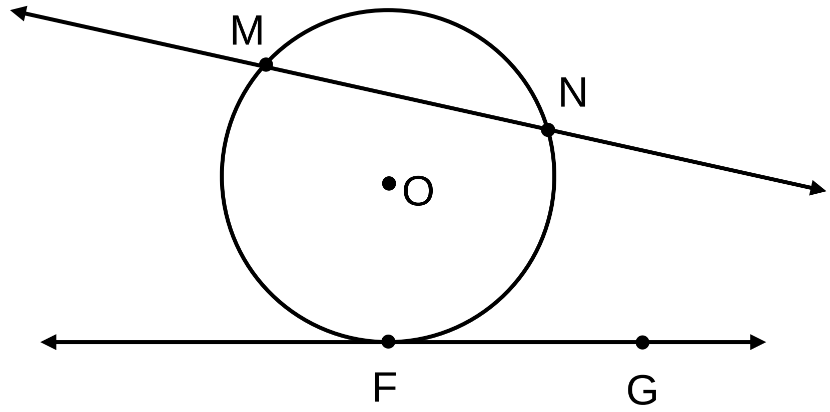
Solution:

Given: Center K is equidistant from the sides of inscribed ΔABC

Prove: ΔABC is equilateral



Statement	Reason
1. Center K is equidistant from the sides of inscribed ΔABC	1. Given
2. KF = KE = KD	2. Def. of equidistant
3. $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{AC}$, $\overline{AB} \cong \overline{AC}$	3. 2 chords equidistant from center are ≅
4. ΔABC is equilateral	4. Def. of equilateral
5. If the center of a circle is equidistant from the sides of an inscribed Δ, the Δ is equilateral	5. Law of Deduction



$\overleftrightarrow{MN}$ is a secant

$\overleftrightarrow{FG}$ is a tangent

F is the point of tangency

Definitions:

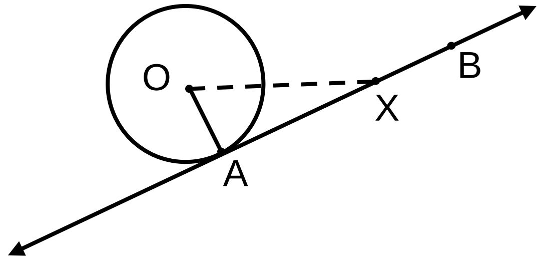
secant – a line that is in the same plane as the circle and intersects the circle in exactly two points.

tangent – a line that is in the same plane as the circle that intersects a circle in exactly one point.

point of tangency – the point at which the tangent intersects the circle

Theorem 9.4: If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.

Indirect Proof:



Given: $\overleftrightarrow{AB}$ is a tangent to circle O at point A

Prove: $\overleftrightarrow{OA} \perp \overleftrightarrow{AB}$

Suppose that $\overleftrightarrow{OA}$ is not perpendicular to $\overleftrightarrow{AB}$. Then there must be some other $\overleftrightarrow{OX}$ that is perpendicular to $\overleftrightarrow{AB}$ (where x 0 $\overleftrightarrow{AB}$). Since we know that the shortest distance from a point to a line is the perpendicular distance, then OX < OA. This implies that X is in the interior of circle O, thus making $\overleftrightarrow{AB}$ a secant, which intersects a circle in two points. But this is a contradiction of the given information that $\overleftrightarrow{AB}$ is a tangent. Thus the assumption that $\overleftrightarrow{OA}$ is not perpendicular to $\overleftrightarrow{AB}$ must be false. Hence $\overleftrightarrow{OA} \perp \overleftrightarrow{AB}$.

Main Steps in an Indirect Proof:

1. Assume the opposite of what you are trying to prove.
2. Reason deductively from the assumption.
3. Reason to a conclusion that contradicts the assumption, the given, or some theorem.
4. Conclude that the assumption is false and therefore the statement you are trying to prove is true.

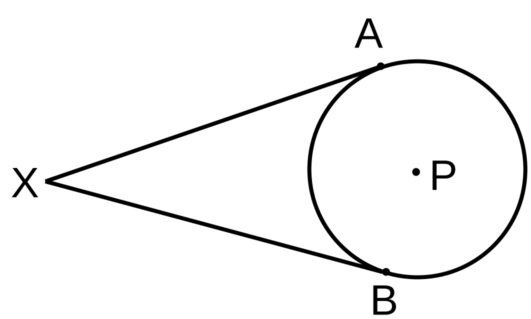
Law of Contradiction (Theorem 9.5): If an assumption leads to a contradiction, then the assumption is false and its negation is true.

Two-column proof of Theorem 9.4 using the Law of Contradiction

Statement	Reason
1. $\overleftrightarrow{AB}$ is tangent to circle O at point A	1. Given
2. $\overleftrightarrow{AB}$ intersects circle O in exactly one point	2. Def. of tangent
3. Assume OA is not perpendicular to AB	3. Assumption
4. Draw the line ⊥ to $\overleftrightarrow{AB}$ that passes through O; let X be the point of intersection	4. Auxiliary line
5. OX < OA	5. Longest Side Inequality (∠X is rt.)
6. X is interior to circle O	6. Def. of interior
7. $\overleftrightarrow{AB}$ contains a chord of circle O	7. Chord Postulate
8. $\overleftrightarrow{AB}$ intersects circle O in 2 points	8. Def. of chord
9. $\overleftrightarrow{OA} \perp \overleftrightarrow{AB}$	9. Law of Contradiction

Theorem 9.6: If a line is perpendicular to a radius at a point on the circle, then the line is tangent to the circle.

Look at:

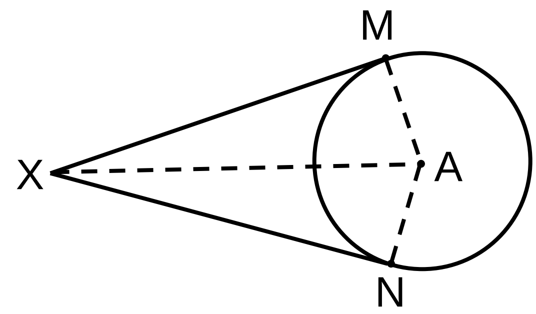


*If you are given a circle and a point in the exterior of the circle, you can find 2 line segments that are tangent to the circle.

Theorem 9.7: Tangent segments extending from a given exterior point to a circle are congruent.

Given: Circle A with exterior point X
$\overline{XM}$ and $\overline{XN}$ are tangent to circle A

Prove: $\overline{XM} \cong \overline{XN}$

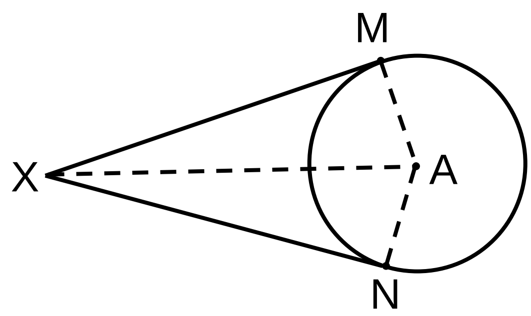


Statement	**Reason**
1. Circle A with exterior point X; $\overline{XM}$ and $\overline{XN}$ are tangent to circle A	1. Given
2. Draw $\overleftrightarrow{AM}$, $\overleftrightarrow{AN}$, $\overleftrightarrow{AX}$	2. Auxiliary lines
3.	3.
4.	4.
5.	5.
6.	6.
7.	7.
8.	8.
9.	9.

Solution:

Given: Circle A with exterior point X
$\overline{XM}$ and $\overline{XN}$ are tangent to circle A

Prove: $\overline{XM} \cong \overline{XN}$

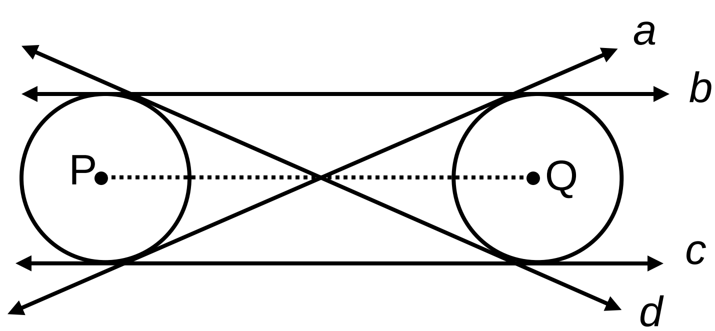


Statement	Reason
1. Circle A with exterior point X; $\overline{XM}$ and $\overline{XN}$ are tangent to circle A	1. Given
2. Draw $\overleftrightarrow{AM}$, $\overleftrightarrow{AN}$, $\overleftrightarrow{AX}$	2. Auxiliary lines
3. $\overline{AM} \cong \overline{AN}$	3. Radii of circle ≅
4. $\overleftrightarrow{AM} \perp \overleftrightarrow{XM}$, $\overleftrightarrow{AN} \perp \overleftrightarrow{XN}$	4. radius to the point of tangency is ⊥ the tangent segment (9.4)
5. ∠XMA and ∠XNA are rt. angles	5. Def. of ⊥
6. ΔXMA and ΔXNA are rt. triangles	6. Def. of right Δ's
7. $\overline{AX} \cong \overline{AX}$	7. Reflexive
8. ΔXAM ≅ ΔXAN	8. HL
9. $\overline{XM} \cong XN$	9. Def. of ≅ Δ's

Definition: A common tangent is a line that is tangent to each of two coplanar circles.

2 Types of Common Tangents:

1. Internal common tangents intersect the segment joining the centers.
2. External common tangents do not intersect the segment joining the centers.

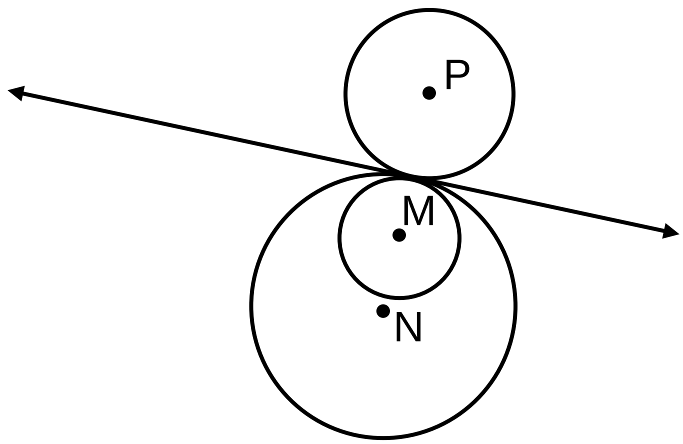


lines *a* and *d* are internal common tangents
lines *b* and *c* are external common tangents

Definition: Tangent circles are coplanar circles that are tangent to the same line at the same point.

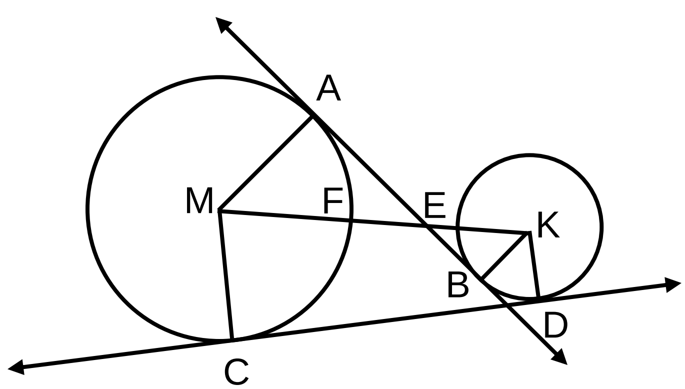
Two Ways that Circles can be Tangent:

1. Internally tangent circles are tangent on the same side of the common tangent.
2. Externally tangent circles are tangent circles on opposite sides of the common tangent.



Circle P and circle N are externally tangent circles.
Circle P and circle M are externally tangent circles.
Circle M and circle N are internally tangent circles.

Sample Problems: Given circle M and circle K with common tangents $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, m∠BKE = 45°, KD = 8, MC = 12



1. Find EB.

ΔEKB is rt. isos. Δ, so EB=BK=8, since radius=8

2. Find AE.

Since ΔEKB is rt. isos. Δ, m$\angle$KEB = 45
Then m$\angle$AEM = 45 (vertical angle with $\angle$KEB)
ΔMAE is rt. isos Δ since $\angle$A is rt. ($\perp$ radius to tan)
AE = MA = 12, since radius = 12

3. Find EF.

<u>Plan</u>: Find ME and subtract MF, which is 12. By Pythag. Thm, ME = 12$\sqrt{2}$ and EF = 12$\sqrt{2}$ - 12

4. Find AB.

AB = AE + EB = 12 + 8 = 20

5. Find MK.

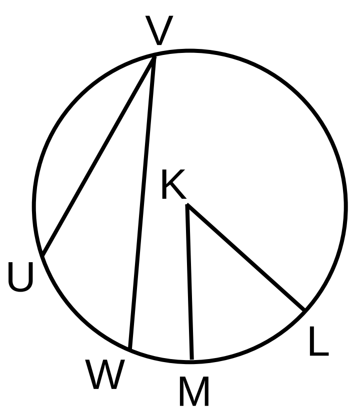
MK = ME + EK	*EK2=EB2 + BK2*
MK = 12$\sqrt{2}$ + 8$\sqrt{2}$	*EK2=82 + 82*
MK = 20$\sqrt{2}$	*EK = 8$\sqrt{2}$*

Definitions:

A central angle of a circle intersects the circle in two points and has its vertex at the center of the circle. (The angle and circle are coplanar.)

An inscribed angle of a circle is an angle with its vertex on a circle and with its sides containing chords of a circle.

***Note**: Each of these angles determines a pair of arcs of the circle.

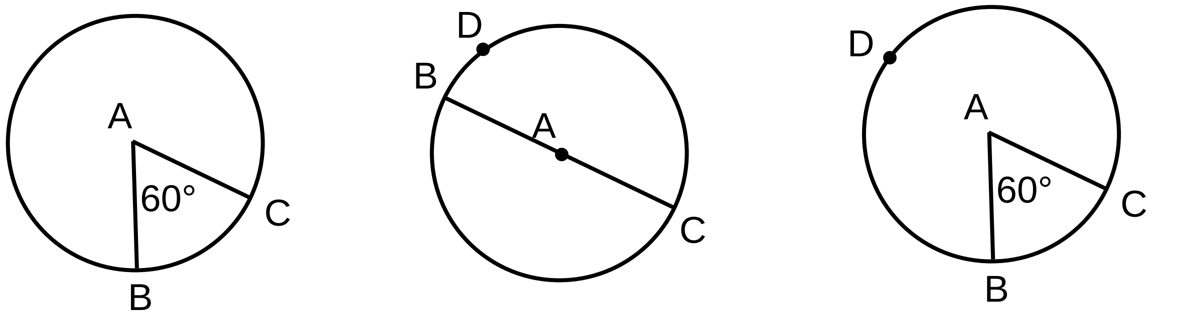


Given: Circle K

$\angle UVW$ is an inscribed angle

$\angle LKM$ is a central angle

<u>Definition</u>: <u>Arc measure</u> is the same measure as the degree measure of the central angle that intercepts the arc.



$m\angle BAC = 60°$
$m\overset{\frown}{BC} = 60°$

$m\angle BAC = 180°$
$m\overset{\frown}{BDC} = 180°$

$m\overset{\frown}{CDB} = 300°$

Definitions:

A <u>minor arc</u> is an arc measuring less than 180°. Minor arcs are denoted with 2 letters, such as $\overset{\frown}{AB}$ where A and B are endpoints of the arc.

A <u>major arc</u> is an arc measuring more than 180°. Major arcs are denoted with 3 letters, such as $\overset{\frown}{ABC}$, where A and C are endpoints and B is another point on the arc.

A <u>semicircle</u> is an arc measuring 180°.

Arc Addition Postulate (9.2): If B is a point on $\overset{\frown}{AC}$, then $m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{AC}$

**The degree measure of a major arc can be given in terms of its associated minor arc because the 2 arcs must add to give the entire circle, which is 360°.

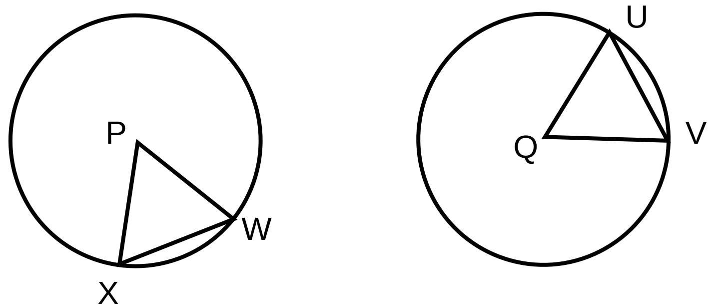
<u>Major Arc Theorem</u> (9.8): $m\overset{\frown}{ACB} = 360° - m\overset{\frown}{AB}$

Definition: Congruent arcs are arcs on congruent circles that have the same measure.

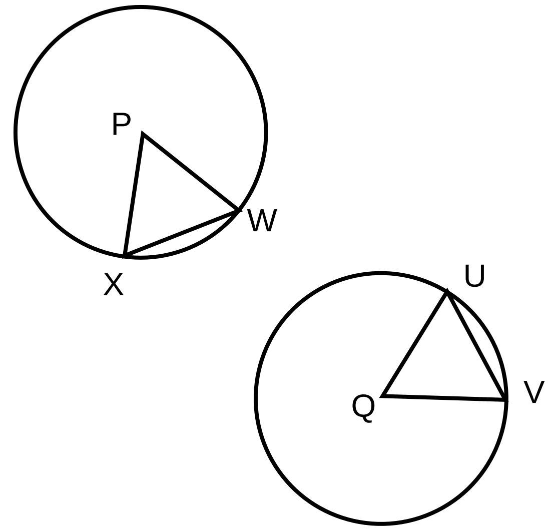
Question: Why do the circles have to be congruent?

Answer: Arcs that have the same degree measure are not the same size if the circles are not the same.

Theorem 9.9: Chords of congruent circles are congruent if and only if they subtend congruent arcs.



Proof of Part 1:



Given: $\odot P$ with chord $\overline{WX}$,
$\odot Q$ with chord $\overline{UV}$,
$\odot P \cong \odot Q$
$\overline{UV} \cong \overline{WX}$

Prove: $\overset{\frown}{UV} \cong \overset{\frown}{WX}$

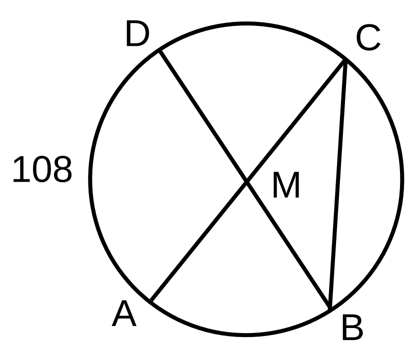
Statement	Reason
1. $\odot P$ with chord $\overline{WX}$, $\odot Q$ with chord $\overline{UV}$, $\odot P \cong \odot Q$, $\overline{UV} \cong \overline{WX}$	1. Given
2. $\overline{QU} \cong \overline{PX}$, $\overline{QV} \cong \overline{PW}$	2. Radii of circles are $\cong$
3. $\Delta UQV \cong \Delta XPW$	3. SSS
4. $\angle UQV \cong \angle XPW$	4. Def. of $\cong$ Δ's
5. $m\angle UQV = m\angle XPW$	5. Def. of $\cong$ $\angle$'s
6. $m\angle UQV = m\overset{\frown}{UV}$, $m\angle XPW = m\overset{\frown}{WX}$	6. Def. of arc measure
7. $m\overset{\frown}{UV} = m\overset{\frown}{XW}$	7. Substitution
8. $\overset{\frown}{UV} \cong \overset{\frown}{XW}$	8. Def. of $\cong$ arcs
9. If $\overline{UV} \cong \overline{WX}$, then $\overset{\frown}{UV} \cong \overset{\frown}{WX}$	9. Law of Deduction

Theorem 9.10: In congruent circles, chords are congruent if and only if the corresponding central angles are congruent.

Theorem 9.11: In congruent circles, minor arcs are congruent if and only if their corresponding central angles are congruent.

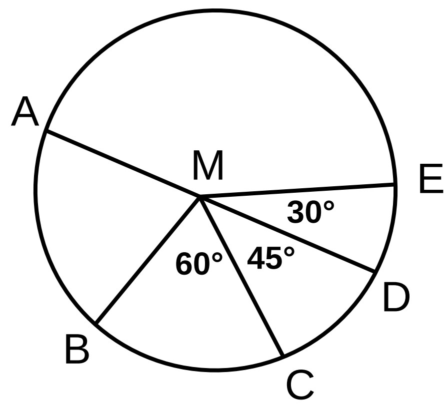
Theorem 9.12: In congruent circles, two minor arcs are congruent if and only if the corresponding major arcs are congruent.

Sample Problem: Given circle M with diameters $\overline{DB}$ and $\overline{AC}$, and $m\overset{\frown}{AD} = 108°$.



1. Find $m\angle AMB$. $m\angle AMB = 180° – 108° = 72°$
2. Find $m\angle BMC$. $m\angle BMC = 108°$ (vert. $\angle$'s)
3. Find $m\overset{\frown}{DAB}$. 180°, since it is a semicircle
4. Find $m\overset{\frown}{DC}$. $m\overset{\frown}{DC} = 180° – 108° = 72°$

Sample Problems: Use the diagram to answer the following questions. $\overline{AD}$ is a diameter.



1. Name 9 minor arcs.

   $\overset{\frown}{AB}$, $\overset{\frown}{AC}$, $\overset{\frown}{BC}$, $\overset{\frown}{BD}$, $\overset{\frown}{BE}$, $\overset{\frown}{CD}$, $\overset{\frown}{CE}$, $\overset{\frown}{DE}$, $\overset{\frown}{AE}$

2. Name 3 major arcs.

   $\overset{\frown}{ACE}$, $\overset{\frown}{ADC}$, $\overset{\frown}{ADB}$, etc.

3. Find $m\overset{\frown}{AB}$.

   $m\angle AMB = 180° – 45° – 60° = 75°$, so $m\overset{\frown}{AB} = 75°$

4. Find $m\overset{\frown}{AE}$.

   $m\angle AME = 180° – 30° = 150°$, so $m\overset{\frown}{AE} = 150°$

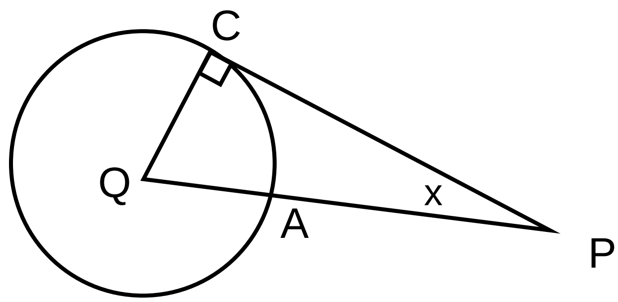
5. Find $m\overset{\frown}{CD} + m\overset{\frown}{DE}$.

   $45° + 30° = 75°$

Sample Problem:

Given: Circle Q with tangent $\overleftrightarrow{CP}$, $m\angle CPQ = x$

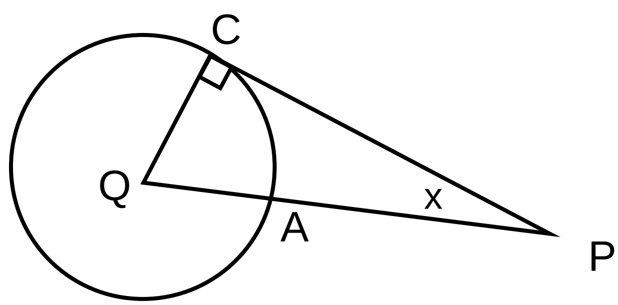
Prove: $m\widehat{AC} = 90° - x$



Statement	**Reason**
1. Circle Q with tangent $\overleftrightarrow{CP}$, $m\angle CPQ = x$	1. Given
2.	2.
3.	3.
4.	4.
5.	5.
6.	6.
7.	7.
8.	8.
9.	9.

Solution:

Given: Circle Q with tangent $\overleftrightarrow{CP}$,
m$\angle$ CPQ = x



Prove: m$\overset{\frown}{AC}$ = 90° - x

Statement	Reason
1. Circle Q with tangent $\overleftrightarrow{CP}$, m$\angle$ $\overset{\frown}{CPQ}$ = x	1. Given
2. $\overline{CQ} \perp \overleftrightarrow{CP}$	2. Radius $\perp$ to tangent (Thm. 9.4)
3. $\angle$QCP is a rt. $\angle$	3. def. of perpendicular
4. Δ QCP is a rt. Δ	4. def. of rt. Δ
5. $\angle$CQP and $\angle$CPQ are complementary	5. acute $\angle$'s of rt. Δ are complementary (6.18)
6. m$\angle$CQA + x = 90°	6. Def. of comp. $\angle$'s
7. m$\angle$CQA = 90° - x	7. Ad. Prop. of Eq.
8. m$\angle$CQA = m$\overset{\frown}{AC}$	8. Def. of arc meas.
9. m$\overset{\frown}{AC}$ = 90° - x	9. Subst. (6 into 7)