Linear Equations in Two Variables

Using Slope

The simplest mathematical model for relating two variables is linear equation in two variables. It is called a linear equation because its graph is a line.

The Slope-Intercept Form of the Equation of a Line

The graph of the equation

$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

(If you let $x = 0$, you get $y = b$, which means the y-intercept of the line is $(0, b)$.)

The value of m tells us the slope of the line. The slope is the steepness of the line. It is the ratio of change in y to the change in x.

- The larger the absolute value of the slope, the steeper the line.

- A line with a positive slope rises from left to right.
- A line with a negative slope falls from left to right.
- A horizontal line has no slope.
- A vertical line has an undefined slope.

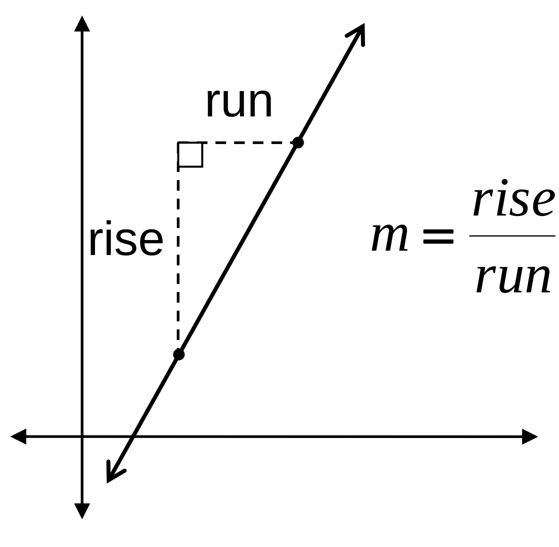
Determining the Slope of a Line

Definition of Slope

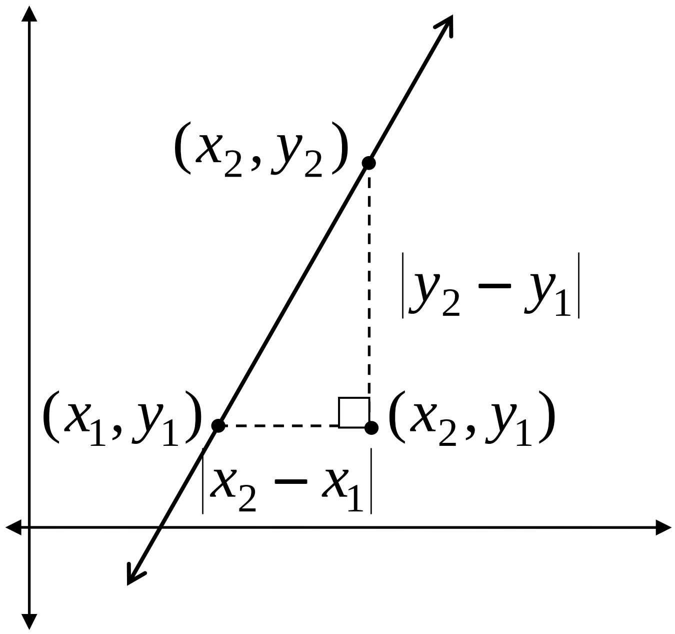
The slope m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \neq x_2$.



If we do not know the slope, we can find it using any two points. Subtracting the y's will give us the rise, and subtracting the x's will give us the run.



$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

*It does not matter which point you start with, as long as you use the same point for the first numbers of the numerator and denominator.

Example: Find the slope of the line containing the points (4, 3) and (2, -5).

<u>*Solution*</u>:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-3}{2-4} = \frac{-8}{-2} = 4$$

Example: Find the slope of the line containing the points (2, -9) and (6, -9).

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-9-(-9)}{6-2} = \frac{0}{4} = 0$$

This line is horizontal. Its equation is y = -9.

Example: Find the slope of the line containing the points (5, 4) and (5, -8).

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-8-4}{5-5} = \frac{-12}{0} = undefined$$

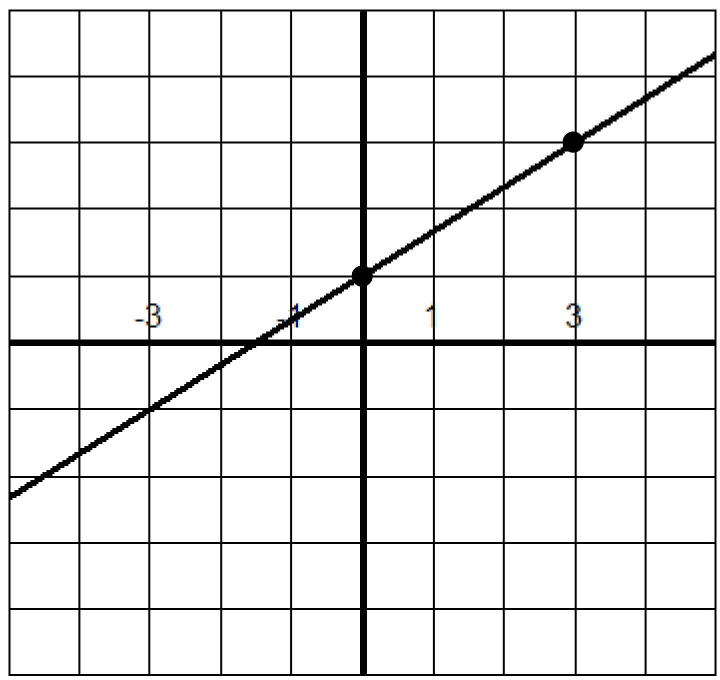
This line is vertical. Its equation is x = 5.

Graphing Linear Equations

To graph a linear equation, first graph the y-intercept and then use the slope to find another point. Draw the line through these two points.

Example: Sketch the graph of $y = \frac{2}{3}x + 1$.

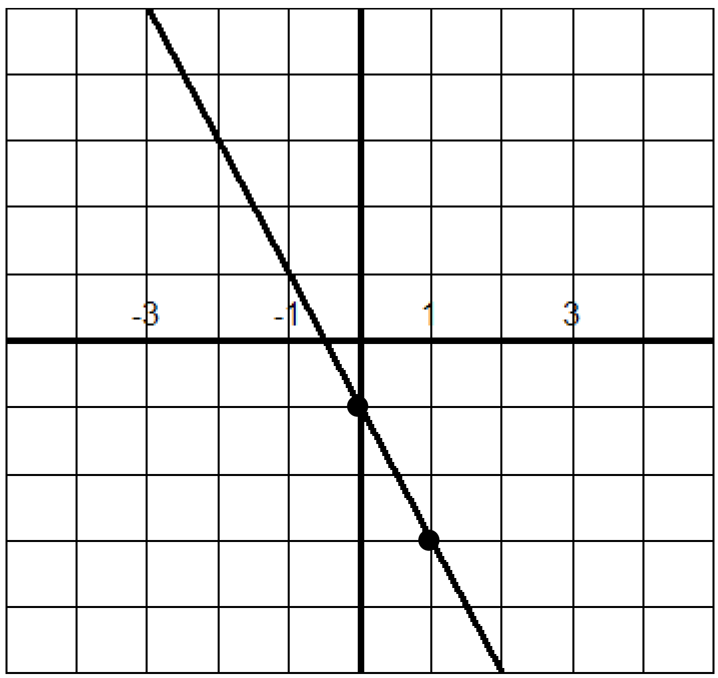
Solution: Plot the y-intercept (0, 1). From this point go up 2 and to the right 3 units. (The rise is 2 and the run is 3.) Draw the line through these two points.



Note: The graphing calculator does not put arrows on the end of any lines, curves, axes, or asymptotes. These lines do, however, go on forever, and when drawing them by hand, should have arrows drawn to show this.

Example: Sketch the graph of $y = -2x - 1$.

Solution: Plot the y-intercept (0, -1). Since the slope is -2, think of it as $\frac{-2}{1}$. This means the rise is -2 and the run is 1. To rise -2, we actually "fall" 2 units. So, from the point (0, -1), go down 2 units and to the right 1 unit. Draw the line through these two points.



Writing Linear Equations in Two Variables

There are 2 forms of equations that are sometimes used to find the equations of lines, depending on the information given. They are the point-slope form and the two-point form.

Point-Slope Form of the Equation of a Line

The equation of the line with slope m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Two-Point Form of the Equation of a Line

The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

An alternative way of finding the equation of a line uses only the slope-intercept form.

Example: Find the equation of the line with slope 4 that passes through the point (-6, 2).

Solution: We start with the equation $y = mx + b$.

- We know that our equation must be $y = 4x + b$.
- Since the line passes through the point (-6, 2), then this point must work in the equation. Plug in the point and solve for b.

Plug in the point (-6, 2) and solve for b.

$$y = 4x + b$$
$$2 = 4(-6) + b$$
$$2 = -24 + b$$
$$b = 26$$

- Replace the value of b into the equation to get

$$y = 4x + 26$$

Example: Find the equation of the line that passes through the points (-2, 2) and (4, 5).

<u>*Solution*</u>: Start with the equation $y = mx + b$.

- We always need the slope, so find the slope using the 2 points.

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-5}{-2-4} = \frac{-3}{-6} = \frac{1}{2}$$

- Use either one of the points in the equation $y = \frac{1}{2}x + b$ to find b.

Plug in the point (4, 5) and solve for b.

$$y = \frac{1}{2}x + b$$
$$5 = \frac{1}{2}(4) + b$$
$$5 = 2 + b$$
$$b = 3$$

- The equation is $y = \frac{1}{2}x + 3$

Example: Find the equation of the line that passes through the points (5, 1) and (-1, 3).

Solution: Start with the equation $y = mx + b$.

- We always need the slope, so find the slope using the 2 points.

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-1-5} = \frac{2}{-6} = \frac{-1}{3}$$

- Use either one of the points in the equation $y = \frac{-1}{3}x + b$ to find b. We will use (5, 1).

$$y = \frac{-1}{3}x + b$$

$$1 = \frac{-1}{3}(5) + b$$

$$1 = \frac{-5}{3} + b$$

Multiply through by 3 to clear the fractions.

$$3 = -5 + 3b$$

$$8 = 3b$$

$$b = \frac{8}{3}$$

- The equation for the line must be $y = \frac{-1}{3}x + \frac{8}{3}$.

Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$
6. Two-Point form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Parallel and Perpendicular Lines

The slopes of 2 nonvertical lines can be used to determine whether the lines are parallel or perpendicular.

Parallel and Perpendicular Lines

Two distinct nonvertical lines are parallel if and only if their slpes are equal. That is, $m_1 = m_2$.

Two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals of each other. That is, $m_1 = \frac{-1}{m_2}$. (Their product = -1)

Example: Are the lines $y = \frac{2}{3}x + 6$ and $y = \frac{2}{3}x + 7$ parallel or perpendicular?

Solution: parallel

Example: What is the slope of a line perpendicular to the line $y = \frac{2}{3}x + 6$?

Solution: -3/2

Example: Find the equation of the line that passes through the point (-3, 1) and is parallel to the line given by $2x + 3y = 1$

Solution: First put the equation in slope-intercept form to determine the slope.

$$2x + 3y = 1$$

$$3y = -2x + 1$$

$$y = \frac{-2}{3}x + \frac{1}{3}$$

The line through the point (-3, 1) must have the same slope. Find b.

$$y = \frac{-2}{3}x + b$$

$$1 = \frac{-2}{3}(-3) + b$$

$$1 = 2 + b$$

$$b = -1$$

The line is $y = \frac{-2}{3}x - 1$.

Example: Find the equation of the line perpendicular to the line $y = \frac{-2}{3}x - 1$ and passes through the point (4, 9). Give the answer in general form.

Solution: The slope must be the negative reciprocal of $\frac{-2}{3}$, which is $\frac{3}{2}$. Use this slope and the point to find b.

$$y = \frac{3}{2}x + b$$

$$9 = \frac{3}{2}(4) + b$$

$$9 = 6 + b$$

$$b = 3$$

The equation is $y = \frac{3}{2}x + 3$. Put this in general form.

$$y = \frac{3}{2}x + 3$$

$$2y = 3x + 6$$

$$3x - 2y + 6 = 0$$

Application

In real-life problems, the slope of a line can be interpreted as either a *ratio* or a *rate*.

- If the x-axis and y-axis have the same unit of measure, then the slope has no units and is a ratio.
- If the x-axis and y-axis have different units of measure, then the slope is a rate or rate of change.

Example: A boat ramp is built so that the ramp rises 24 inches over a horizontal length of 20 feet. What is the slope of the ramp?

Solution: Since we can change the inches into feet so that the measurements are the same, we write the slope as

$$m = \frac{rise}{run} = \frac{2\,ft.}{20\,ft.} = \frac{1}{10}$$

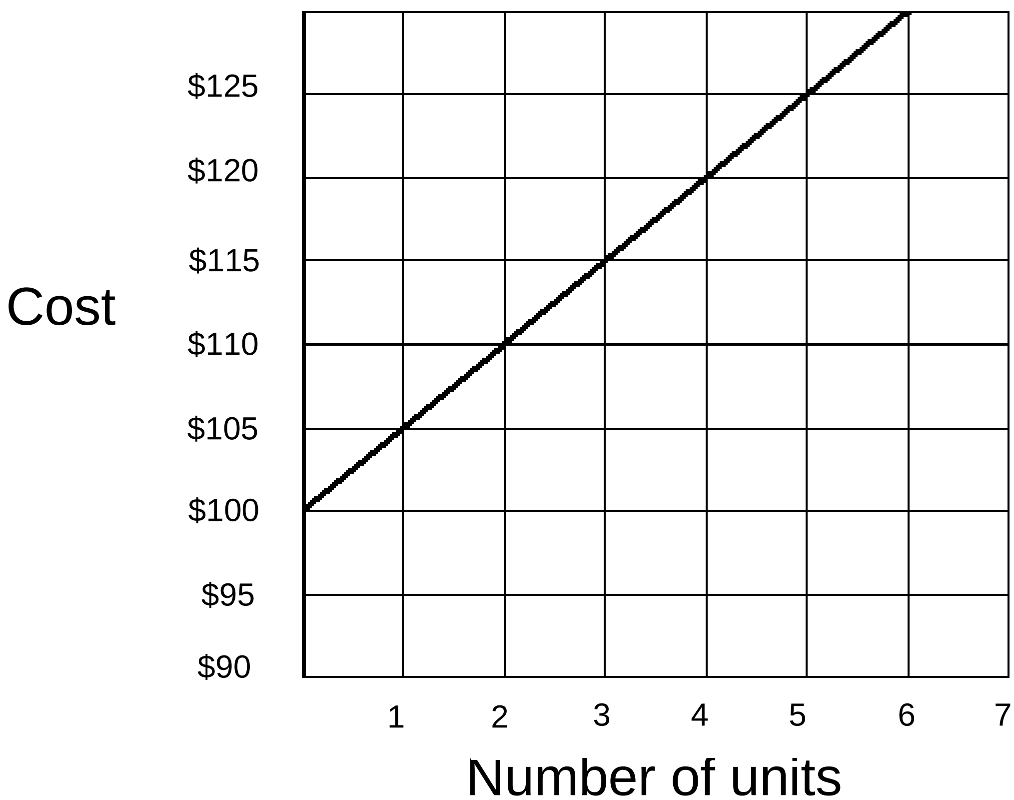
(Notice that the measurements "cancel" each other and we are left with the slope with no unit measure.)

Example: A company that manufactures clocks determines the cost of producing x toys is

$$C = 5x + 100$$

Describe the practical significance of the y-intercept and the slope of this line.

Solution: Look at the graph



The y-intercept (0, 100) tells us that even if no units are produced, the cost is $100. This fixed cost could be things like rent or machines. The slope tells us the cost of producing each unit.

If the cost of depreciation is the same amount every year, then we call this linear or straight-line depreciation.

Example: A company purchases a $20,000 machine. In 4 years the machine will be worth $10,000. Write a linear equation that relates the value V of the machine after t year.

Solution: Think of this in terms of points. At 0 years, the value of the machine is $20,000, so we have the point (0, 20,000). After 4 years the value is $10,000, so the point is (4, 10,000). Find the slope between these 2 points.

$$m = \frac{\Delta y}{\Delta x} = \frac{20{,}000 - 10{,}000}{0 - 4} = -2500$$

Find the equation:

$$y = mx + b$$

$$V = -2500t + b$$

Put in one of the points to find b.

$$V = -2500t + b$$
$$20{,}000 = -2500(0) + b$$
$$b = 20{,}000$$

Therefore the equation must be

$$V = -2500t + 20{,}000$$

Intercept Form of an Equation

If an equation is of the form $\frac{x}{a} + \frac{y}{b} = 1$,
then the x-intercept is a and the y-intercept is b.

Example: Find the intercepts of $\frac{x}{3} + \frac{y}{-5} = 1$

Solution: The x-intercept is (3, 0) and the y-intercept is (0, -5).

Note: If we have to find the standard form we need to clear the fractions by multiplying through by 15.