# Shifting, Reflecting, and Stretching Graphs

## Shifting Graphs

Graph $y_1 = x^2$ This is $f(x) = x^2$
Graph $y_2 = (x-4)^2$ This is $f(x-4)$
Graph $y_3 = (x+4)^2$ This is $f(x+4)$

*What happens to the graph?*

$f(x-4)$ is $f(x)$ shifted 4 units to the right.
$f(x+4)$ is $f(x)$ shifted 4 units to the left.

Turn off $y_2$ and $y_3$.

Graph $y_4 = x^2 + 4$ This is $f(x)+4$
Graph $y_5 = x^2 - 4$ This is $f(x)-4$

*What happens to the graph?*

$f(x)+4$ is $f(x)$ shifted 4 units up.
$f(x)-4$ is $f(x)$ shifted 4 units down.

## Vertical and Horizontal Shifts

Let $c$ be a positive real number. The following changes in the function $y = f(x)$ will produce the stated shifts in the graph of $y = f(x)$

1. $h(x) = f(x - c)$ Horizontal shift *c* units to the right
2. $h(x) = f(x + c)$ Horizontal shift *c* units to the left
3. $h(x) = f(x) - c$ Vertical shift *c* units downward
4. $h(x) = f(x) + c$ Vertical shift *c* units upward

**Example:** Given $f(x) = x^3 + x$, describe the shifts of the graph of *f* generated by the following functions.

**a)** $g(x) = (x + 1)^3 + x + 1$

*Horizontal shift 1 unit to the left.*

**b)** $h(x) = (x - 4)^3 + x – 4$

*Horizontal shift 4 units to the right.*

**Example:** Let *f(x)* = │x│. Write the equation for the function resulting from a vertical shift of 3 units downward and a horizontal shift of 2 units to the right of the graph of *f(x)* = │x│.

*Answer:* *f(x)* = │x -2│ - 3.

<u>Reflecting Graphs</u>

Graph $y_1 = f(x) = (x-2)^3$

Graph $y_2 = f(x) = -(x-2)^3$ Note that this is $-f(x)$.

Graph $y_3 = f(x) = (-x-2)^3$ Note that this is $f(-x)$.

What happens to the graph?

$-f(x)$ *is reflected in the $x$-axis*

$f(-x)$ *is reflected in the $y$-axis*

The following changes in $y = f(x)$ will produce the stated reflections of the graph of $y = f(x)$.

1. $h(x) = -f(x)$: reflection in the $x$-axis
2. $h(x) = f(-x)$: reflection in the $y$-axis

**Example**: Let $f(x)=|x|$ Describe the graph of $g(x)=-|x|$ in terms of $f$.

*The graph of $g$ is a reflection of the graph of $f$ in the $x$-axis.*

**<u>Definition</u>**: A <u>rigid transformation</u> is a transformation in which the basic shape of the graph is unchanged.

Rigid transformations change only the <u>position</u> of the graph in the $xy$-plane.

Three types of rigid transformations:

1. Horizontal shifts
2. Vertical shifts
3. Reflections

## <u>Nonrigid Transformations</u>

Graph $y_1 = f(x) = x^2$

Graph $y_2 = f(x) = 8x^2$ Note that this is $8 \cdot f(x)$.

Graph $y_3 = f(x) = \frac{1}{4}x^2$ Note that this is $\frac{1}{4} \cdot f(x)$.

What happens to the graph?

$y = 8x^2$ *appears narrower than* $y = x^2$

$y = \frac{1}{4}x^2$ *appears flatter than* $y = x^2$

**Definition:** A non-rigid transformation is a transformation that actually distorts the shape of a graph instead of just shifting or reflecting it.

From our example:

$y = 8x^2$ is called a vertical stretch

$y = \frac{1}{4}x^2$ is called a vertical shrink

Graph $y_1 = f(x) = x^3$

Graph $y_2 = f(x) = (2x)^3$ Note that this is $f(2x)$.

Graph $y_3 = f(x) = (\frac{1}{4}x)^3$ Note that this is $f(\frac{1}{4}x)$.

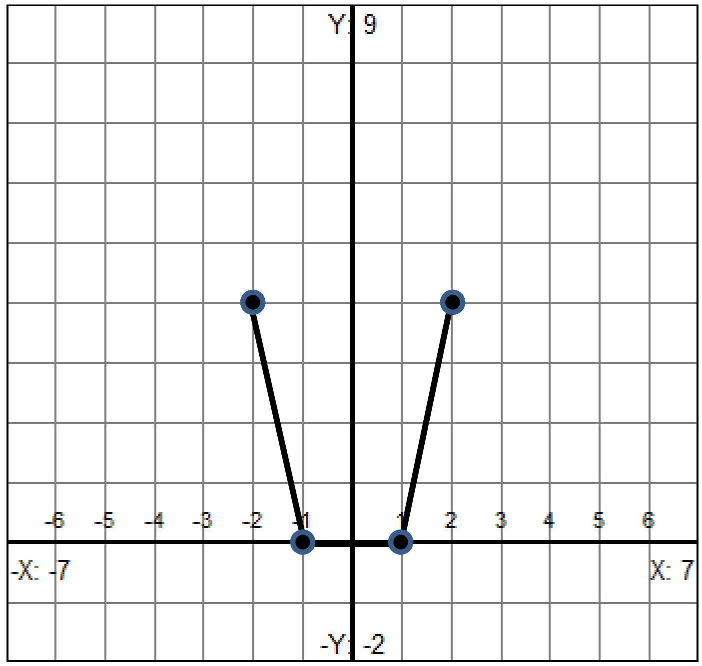
What happens to the graph?

$y = (2x)^3$ *appears steeper than* $y = x^3$

$y_3 = (\frac{1}{4}x)^3$ *appears wider than* $y = x^3$

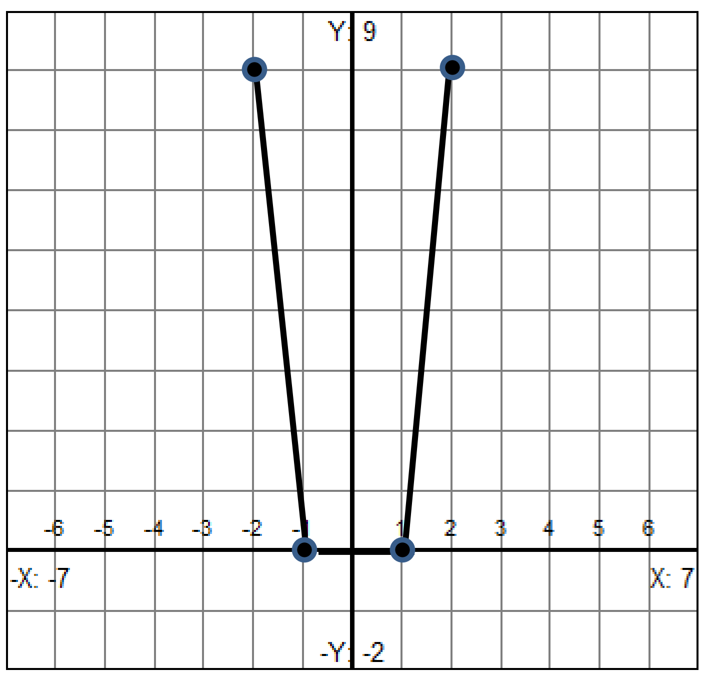
# Consider the graph of *f(x):*

| *x* | *f(x)* |
|---|---|
| -2 | 4 |
| -1 | 0 |
| 1 | 0 |
| 2 | 4 |




## Now look at *2f(x):*

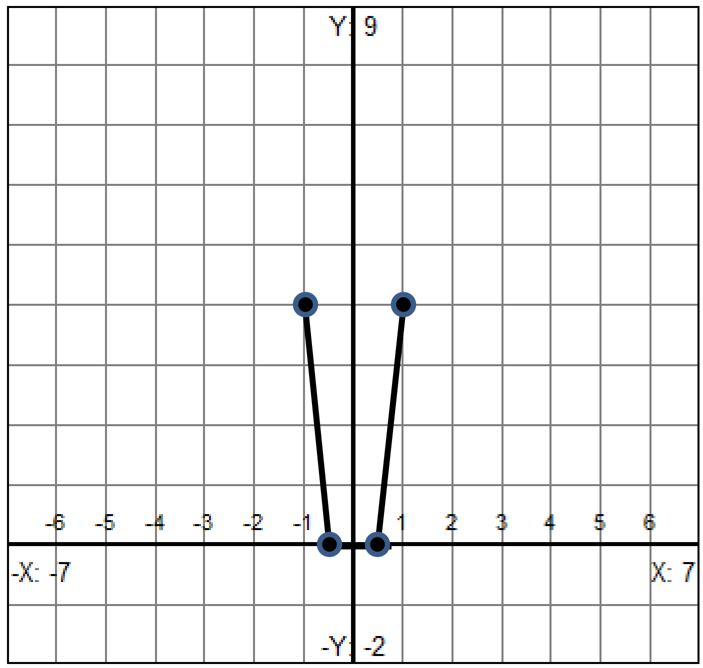
| x | 2f(x) |
|---|---|
| -2 | 8 |
| -1 | 0 |
| 1 | 0 |
| 2 | 8 |



This is a vertical stretch by a factor of 2. Notice that the horizontal aspect of the graph has not changed.

Now look at *f(2x):*

| x | f(2x) |
|---|---|
| -1 | 4 |
| -½ | 0 |
| ½ | 0 |
| 1 | 4 |




This is a horizontal shrink by a factor of 2. Notice that the vertical aspect of the graph has not changed.

In general, for $y = f(x)$ and the real number $c$,

- A <u>vertical stretch</u> is written $g(x) = cf(x)$, where $c > 1$
- A <u>vertical shrink</u> is written $g(x) = cf(x)$, where $0 < c < 1$
- A <u>horizontal shrink</u> is written $h(x) = f(cx)$, where $c > 1$
- A <u>horizontal stretch</u> is written $h(x) = f(cx)$, where $0 < c < 1$

**Example**: Compare the graph of each function with the graph of $f(x) = 4 + x^2$.

**(a)** $g(x) = f(2x)$

$$g(x) = f(2x) = 4 + (2x)^2 = 4 + 4x^2$$

*This is a <u>horizontal shrink</u> of the graph of f(x).*

**(b)** $h(x) = f(\frac{1}{3}x)$

$$h(x) = f(\frac{1}{3}x) = 4 + (\frac{1}{3}x)^2 = 4 + \frac{1}{9}x^2$$

*This is a <u>horizontal stretch</u> of the graph of f(x).*

**Example:** Compare each graph with the graph of $f(x) = \sqrt{x}$

**(a)** $g(x) = -\sqrt{x}$

*A reflection of f in the $x$-axis, since g(x) = -f(x).*

**(b)** $h(x) = \sqrt{-x}$

*A reflection of f in the $y$-axis, since h(x) = f(-x).*

**Example:** Compare each graph with the graph of f(x) = $\sqrt{x}$

**(c)** $k(x) = -\sqrt{x+2}$

*A shift of f 2 units left, followed by a reflection in the x-axis, since k(x) = -f(x+2)*

**(d)** $p(x) = \sqrt{3x}$

*A horizontal shrink, since* $p(x) = f(3x)$.

Consider $j(x) = 2\sqrt{x+1} + 3$ compared to the graph of $f(x) = \sqrt{x}$

Graph $y_1 = f(x) = \sqrt{x}$
Graph $y_2 = y_1(x+1)$ (shift the graph 1 unit left)
Graph $y_3 = y_2 + 3$ (shift the graph 3 unit up)
Graph $y_4 = 2y_3$ (vertical stretch of 2)

Graph $y_5 = y_1(x+1)$ (shift the graph 1 unit left)
Graph $y_6 = 2y_5$ (vertical stretch of 2)
Graph $y_7 = y_5 + 3$ (shift the graph 3 unit up)

Turn off all graphs except $y_4$ and $y_7$. The graphs are not the same. To see which is correct,

Graph $y_8 = 2\sqrt{x+1}+3$

Which graph matches the correct answer?

*Answer: $y_7$*

*Conclusion*: Always follow the order of operations when considering transformations.

<u>Summary</u>

In general, for $y = f(x)$ and the real number $c$,

| | |
|---|---|
| $h(x) = f(x-c)$ | Horizontal shift *c* units to the right |
| $h(x) = f(x+c)$ | Horizontal shift *c* units to the left |
| $h(x) = f(x)-c$ | Vertical shift *c* units downward |
| $h(x) = f(x)+c$ | Vertical shift *c* units upward |
| $h(x) = -f(x)$ | Reflection in the $x$-axis |
| $h(x) = f(-x)$ | Reflection in the $y$-axis |
| $h(x) = cf(x)$ | Vertical stretch, where $c > 1$ |
| $h(x) = cf(x)$ | Vertical shrink, where $0 < c < 1$ |
| $h(x) = f(cx)$ | Horizontal shrink, where $c > 1$ |
| $h(x) = f(cx)$ | Horizontal stretch, where $0 < c < 1$ |