Combinations of Functions

Arithmetic Combinations of Functions:

Definition: Just as two real numbers can be combined with arithmetic operations, two functions can be combined by the operations of addition, subtraction, multiplication, and division to create new functions. A combined function like this is called an arithmetic combination of functions.

The **domain** of an arithmetic combination of functions *f* and *g* consists of all real numbers that are common to the domains of *f* *and* *g*. (i.e. their intersection). In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Definition: Let f and g be functions with overlapping domains. Then for all x common to both domains:

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(fg)(x) = f(x) \cdot g(x)$
4. $\left(\dfrac{f}{g}\right)(x) = f(x) \div g(x)$, if $g(x) \neq 0$.

Examples: Given $f(x) = x^2 + 2x$ and $g(x) = 2x + 1$, find the following.

a) $(f+g)(x)$

$$
\begin{aligned}
(f+g)(x) &= f(x) + g(x) \\
&= (x^2+2x) + (2x+1) \\
&= x^2+4x+1
\end{aligned}
$$

b) $(f-g)(x)$

$$
\begin{aligned}
(f-g)(x) &= f(x) - g(x) \\
&= (x^2+2x) - (2x+1) \\
&= x^2 - 1
\end{aligned}
$$

c) $(fg)(x)$

$$
\begin{aligned}
(fg)(x) &= f(x) \bullet g(x) \\
&= (x^2+2x)(2x+1) \\
&= 2x^3 + 5x^2 + 2x
\end{aligned}
$$

d) $\left(\frac{f}{g}\right)(x)$

$$
\begin{aligned}
\left(\frac{f}{g}\right)(x) &= f(x) \,/\, g(x) \\
&= \frac{x^2+2x}{2x+1}
\end{aligned}
$$

$x \neq -½$ (since $x = -½$ would make the denominator = 0)

*Watch your domains. The domain for your arithmetic function is where your 2 domains overlap.

*For quotient functions you also need to consider what values must b excluded because they make the denominator = 0.

Example: Let f(x) = 7x-5 and g(x) = 3-2x. Find (f-g)(4).

$$(f-g)(x) = (7x-5) - (3-2x)$$
$$= 9x - 8$$
$$(f-g)(4) = 9(4) - 8$$
$$(f-g)(4) = 28$$

Note: You could have found $f(4)$ and $g(4)$ separately and then subtracted the results.

To Graph Arithmetic Combinations of Functions on the Graphing Calculator

Let $f(x) = 7x-5$ and $g(x) = \sqrt{x}$

To graph $(f+g)(x)$:

1. Enter $y_1 = 7x-5$ and $y_2 = \sqrt{x}$
2. Enter $y_3 = y_1 + y_2$ for $(f+g)(x)$:To enter y_1 you must press [VARS] [Y-VARS] [function] [y_1]. Do the same for y_2.
3. Highlight only the = for y_3 and press [Graph].

Note: Use the same procedure for $(f - g)(x)$, $(fg)(x)$,
And $(f/g)(x)$.

To find the domain of $(f+g)(x)$ on the graphing calculator:

1. With $(f+g)(x)$ on the graphing screen, press [TRACE].
2. Move the cursor along with the right and left arrows. When the y = at the bottom of the graphing window is empty, the corresponding value of x is not in the domain. Determine the interval(s) over which your function is defined. This is the domain
3. To decide whether the endpoints of the domain intervals are included or not, while tracing, enter the apparent end of the interval and press [ENTER]. If that endpoint is included, you will get a corresponding value for y. If not, then it will be blank.

Note: The domain for $(f+g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $(f/g)(x)$ will be the same with one exception. For $(f/g)(x)$ you must remove from your domain any values that would make the denominator of your function $(f/g)(x) = 0$

Composition of Functions

Definition: The composition of the function *f* with the function *g* is defined as

$$(f \circ g)(x) = f(g(x))$$

The domain of $(f \circ g)(x)$ is the set of all *x* in the domain of *g* such that *g*(*x*) is in the domain of *f*.

(What comes out of g is what goes into f.)

Example: Given $f(x) = x + 3$ and $g(x) = x^2 + 1$

(a) Find $(f \circ g)(2)$

$(f \circ g)(2) = f(g(2)) = f(5)$, *since* $g(2) = 5$
$f(5) = 8$
Therefore, $(f \circ g)(2) = 8$

(b) Find $(g \circ f)(3)$

$(g \circ f)(3) = g(f(3)) = g(6)$, *since* $f(3) = 6$
$g(6) = 37$
Therefore, $(g \circ f)(3) = 37$

Example: Let $f(x) = 3x + 4$ and $g(x) = 2x^2 - 1$.

(a) Find $(f \circ g)(x)$

$$(f \circ g)(x) = f(g(x)) = f(2x^2 - 1) = 3[2x^2 - 1] + 4 = 6x^2 + 1$$

(b) Find $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) = g(f(x)) = g(3x + 4) &= 2[3x + 4]^2 - 1 \\ &= 2(9x^2 + 24x + 16) - 1 \\ &= 18x^2 + 48x + 31\end{aligned}$$

<u>To Graph or Calculate Composite Functions on the Graphing Calculator</u>

Let $f(x) = 7x - 5$ and $g(x) = x^2$

To graph $(f \circ g)(x)$ and $(g \circ f)(x)$:

1. Enter $y_1 = 7x - 5$ and $y_2 = x^2$
2. For $(f \circ g)(x)$, enter $y_3 = y_1(y_2)$. This puts the values coming out of the function g into the function f.
3. For $(g \circ f)(x)$, enter $y_4 = y_2(y_1)$. This puts the values coming out of the function f into the function g.

To calculate values of $(f \circ g)(x)$ and $(g \circ f)(x)$:

After entering the information above, use the [TABLE] feature to find values for $(f \circ g)(x)$ (stored in y_3) and $(g \circ f)(x)$ (stored in y_4).

Applications

Example: The function $f(x) = 0.06x$ represents the sales tax owed on a purchase with a price tag of x dollars and the function $g(x) = 0.75x$ represents the sale price of an item with a price tag of x dollars during a “25% off” sale. Using one of the combinations of functions discussed in this section, write the function that represents the sales tax owed on an item with a price tag of x dollars during a “25% off” sale.

Think: First we find the sale price. Then we take that result and find the sales tax. That's doing g(x) and then f(x), or f(g(x)).

$$f(g(x)) = f(0.75x) = 0.06(0.75x) = 0.045x$$

$$(f \circ g)(x) = 0.045x$$

So, the tax on an item that costs x dollars is .045x.