Inverse Functions

Look at the function f = {(1,5), (2,6), (3, 7), (4, 8)}

If we interchange the coordinates we get:

{(5, 1), (6, 2), (7, 3), (8, 4)}

This is called the <u>inverse function</u> of the function f and we use the notation f^{-1}.

So, f^{-1} = {(5, 1), (6, 2), (7, 3), (8, 4)}

<u>Note</u>: The domain of f is the same as the range for f^{-1}
The range of f is the same as the domain for f^{-1}

<u>Also note</u>: Functions and their inverse tend to "undo" each other.

$f(f^{-1}(6)) = f(2) = 6$ and $f^{-1}(f(3)) = f(7) = 3$

**This gives us a great test for inverses. If f(g(x)) = x and g(f(x)) = x, then we know that the functions are inverses.

Example: Verify that $f(x) = 2x - 3$ and $g(x) = \frac{x+3}{2}$ are inverse functions.

$$f(g(x)) = f\left(\frac{x+3}{2}\right) = 2\left[\frac{x+3}{2}\right] - 3 = (x+3) - 3 = x$$

$$g(f(x)) = g(2x - 3) = \left(\frac{[2x-3]+3}{2}\right) = \frac{2x}{2} = x$$

Since both $f(g(x)) = x$ and $g(f(x)) = x$, the functions are inverses.

The Graph of an Inverse Function

Graph $y_1 = x^3$
Graph $y_2 = \sqrt[3]{x}$ This is the inverse of $y = x^3$

Now graph $y_3 = x$

**Notice that the graph of $y = \sqrt[3]{x}$ is the reflection of $y = x^3$ in the line $y = x$.

Summary: If the point (a, b) lies on the graph of f, then the point (b , a) must lie on the graph of f^{-1} and vice versa. The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

Drawing Inverses on the Graphing Calculator

You can draw the inverse of any equation entered at the [y=] screen but you cannot *graph* it. This means that you will be able to *see* it on your graph, but you cannot use [TRACE] or [2nd] [CALC] or any other operations on it.

To draw the inverse of a function:

1. Enter and equation into Y_1.
2. Press [2nd] [DRAW] [DrawInv]
3. At the home screen, after DrawInv enter Y_1 and press [ENTER].

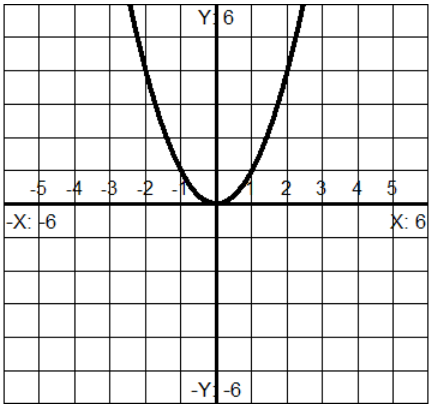
(Reminder: To enter Y_1, press [VARS] [Y-VARS] [Function] [Y_1].)

Note: If you find the inverse of a function algebraically, you check it on your calculator by doing the following:

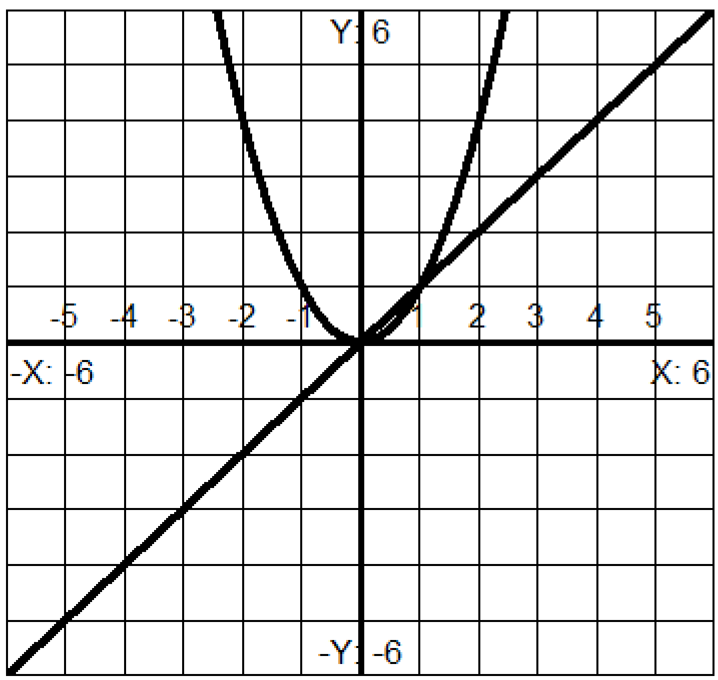
1. Enter your function in Y_1.
2. Enter the inverse that you found in Y_2.
3. Draw the inverse of Y_1.

If the inverse is drawn coincides with Y_2, then your inverse is correct.

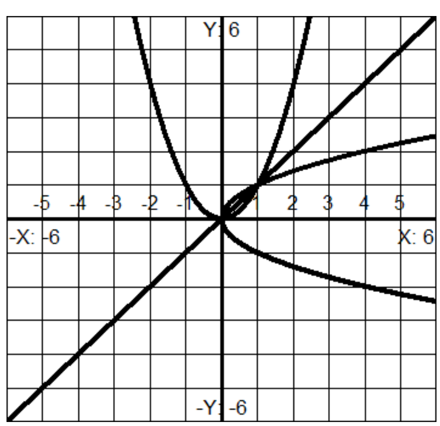
One-to-One Functions



Graph the inverse of this function by reflecting over the line y = x.



Solution:



Is the inverse a function?

no

When is the inverse of a function also a function?

If every y-coordinate of the original function is paired with only one x-coordinate.

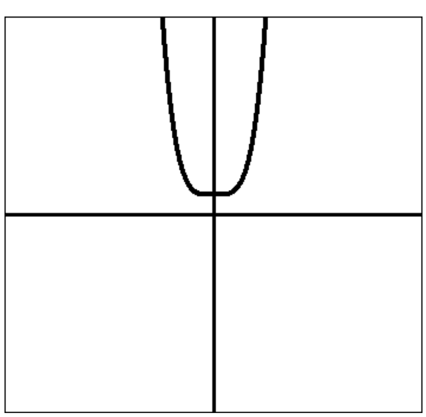
We call this the Horizontal Line Test.

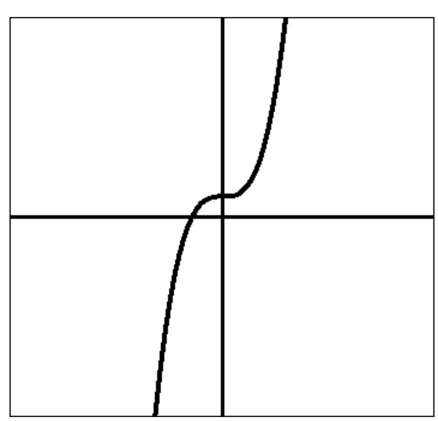
Horizontal Line Test: If every horizontal line meets the graph of a function in at most one point, then the inverse of this function will be a function.

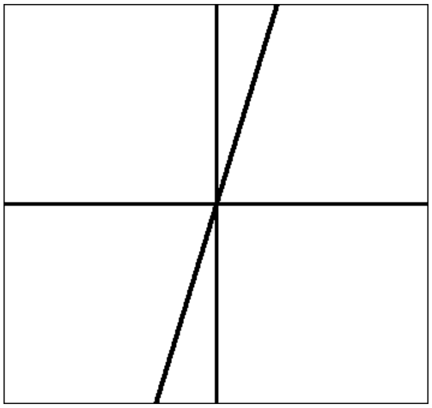
Definition: A function *f* is one-to-one if each value of the dependent variable corresponds to exactly one value of the independent variable. A function has an inverse function if and only if the function is one-to-one.

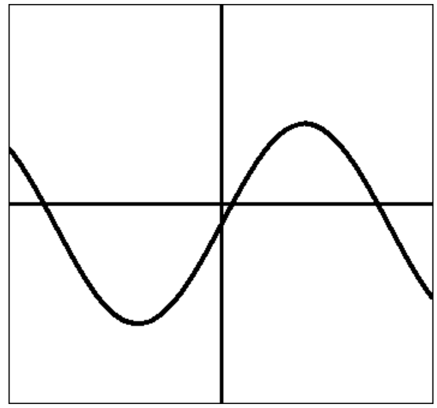
**Every function has an inverse, but the inverse itself is not always a function.

Example: Which of these functions have an inverse function?









Solution: no, yes, yes, no (reading left to right)

<u>Finding Inverse Functions Algebraically</u>

<u>Steps for finding the inverse of a function</u>:

1. Replace $f(x)$ with y.
2. Interchange the roles of x and y.
3. Solve this new equation for y.
4. Replace y by $f^{-1}(x)$.

* <u>Note</u>: To see if your inverse is a <u>function</u>, you will have to do the Horizontal Line Test on the original function, or the Vertical Line Test on your inverse.

To verify that f^{-1} and f are inverse functions:

1. Show that the domain of one is the same as the range of the other, and vice versa.
2. Show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Example: Find the inverse of each of the following functions.

a) $f(x) = \sqrt[3]{x-5}$

$y = \sqrt[3]{x-5}$

$x = \sqrt[3]{y-5}$

$x^3 = y - 5$

$y = x^3 + 5$

$f^{-1}(x) = x^3 + 5$

<u>Verify</u>:

The domain of f = {real #'s}
The range of f = { real #'s}
The domain of f^{-1} = { real #'s}
The range of f^{-1} = { real #'s}

Show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

$f(f^{-1}(x)) = f(x^3 + 5) = \sqrt[3]{[x^3+5]-5} = \sqrt[3]{x^3} = x$

$f^{-1}(f(x)) = f^{-1}(\sqrt[3]{x-5}) = [\sqrt[3]{x-5}]^3 + 5 = x - 5 + 5 = x$

b) $g(x) = \frac{x-4}{x+2}$

$$y = \frac{x-4}{x+2}$$

$$x = \frac{y-4}{y+2}$$

$$x(y+2) = (y-4)$$

$$xy+2x = y-4$$

$$xy-y = -2x-4$$

$$y(x-1) = -2x-4$$

$$y = \frac{-2x-4}{x-1}$$

$$g^{-1}(x) = \frac{-2x-4}{x-1}$$

c)

$$f(x) = 4x - 5$$

$$y = 4x - 5$$

$$x = 4y - 5$$

$$x + 5 = 4y$$

$$4y = x + 5$$

$$y = \tfrac{1}{4}x + \tfrac{5}{4}$$

$$f^{-1}(x) = \tfrac{1}{4}x + \tfrac{5}{4}$$