Lines

We have learned that the graph of a linear equation

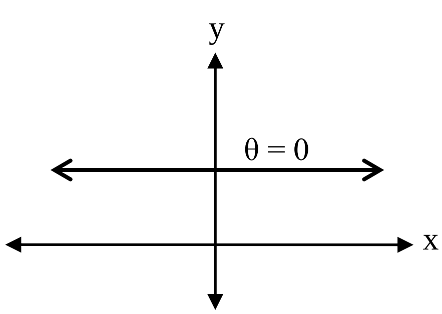
$$y = mx + b$$

is a nonvertical line with slope m and y-intercept $(0, b)$.

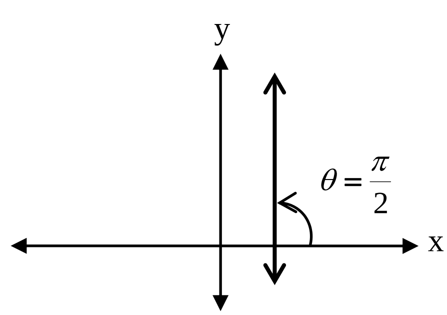
We can also look at the angle that such a line makes with the x-axis. This is called the line's <u>inclination</u>.

Definition of Inclination

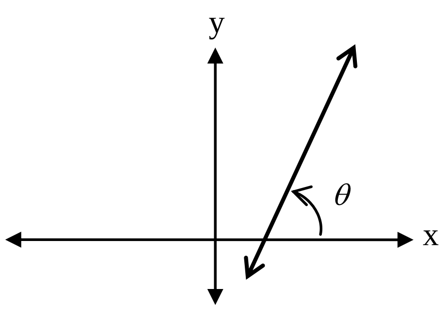
The <u>inclination</u> of a nonhorizontal line is the positive angle θ (less than π) measured counterclockwise from the x-axis to the line.



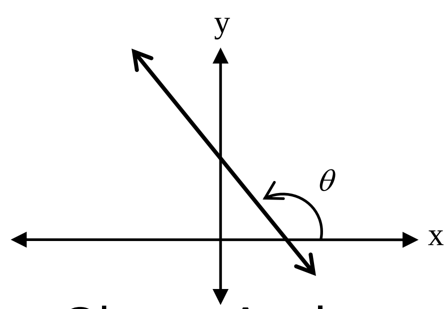
Horizontal Line



Vertical Line



Acute Angle



Obtuse Angle

The inclination of a line is related to its slope in the following way:

> **Inclination and Slope**
>
> If a nonvertical line has inclination θ and slope m, then
>
> $$m = tan\ \theta.$$

Example: Find the inclination of the line $y = \frac{1}{2}x + 5$.

Solution: The slope of the line is $\frac{1}{2}$, so for its inclination,

$$\tan\theta = \frac{1}{2}.$$

Because the slope is positive, we know that the graph goes up and to the right, and thus makes an acute angle with the x-axis. This corresponds to a Quadrant I angle and we can use arctan(½) to find the answer.

$$\theta = \arctan\left(\frac{1}{2}\right) = 26.6°$$

Example: Find the inclination of the line $2x + y = 5$.

Solution: The equation of the line can be written as $y = -2x + 5$, which tells us that slope of the line is -2, so for its inclination,

$$\tan\theta = -2.$$

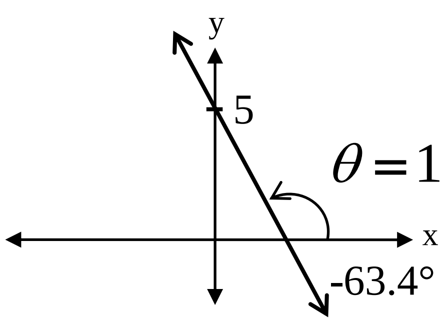
Because the slope is negative, we know that the graph goes down and to the right, and thus makes an obtuse angle with the x-axis. This corresponds to a Quadrant II angle.

If we use arctan to find the answer, our calculator tells us

$$\theta = \arctan(-2) = -63.4°$$

This is because the range for the arctangent function is from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. (*or* -90° to 90°)

Since our angle θ is obtuse (think of a Quadrant II angle), we need to find θ by using 63.4° as the reference angle.

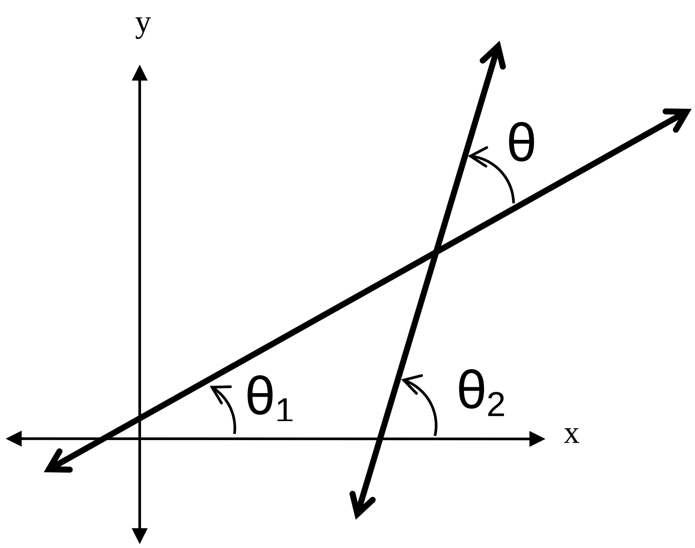


$$\theta = 180° - 63.4° = 116.6°$$

The Angle Between Two Lines

If two distinct lines intersect and are not perpendicular, then their intersection forms two pairs of opposite angles (also called vertical angles). The **smaller** of these angles is called the angle between the two lines.

Look at the graph of the 2 intersecting lines.



Because θ_2 is the exterior angle of the triangle, it must be equal to the sum of the remote interior angles. Thus,

$$\theta + \theta_1 = \theta_2$$

So, $\theta = \theta_2 - \theta_1$ where $\theta_1 < \theta_2$.

Using the formula for the formula of the difference of two angles, we get

$$\theta = \theta_2 - \theta_1$$
$$\tan\theta = \tan(\theta_2 - \theta_1)$$
$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}$$

This is from the formula for tan(u-v)

Since $m = \tan\theta$, we end up with the following formula:

Angle Between Two Lines

If two nonperpendicular lines have slopes m_1 and m_2, the angle between the two lines is

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

*Note: The reason that we use the absolute value brackets is because the angle we are finding is acute, which means its tangent must be positive.

Example: Find the angle between the lines given by
line 1: $3x + 2y = 8$ and line 2: $4x - 5y = 1$.

Solution: Line 1: $y = -\frac{3}{2}x + 4$ $m_1 = -\frac{3}{2}$

Line 2: $y = \frac{4}{5}x - \frac{1}{5}$ $m_2 = \frac{4}{5}$

Use the formula: $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{4}{5} - \frac{-3}{2}}{1 + \left(\frac{4}{5}\right)\left(\frac{-3}{2}\right)}\right| = \left|\frac{\frac{23}{10}}{\frac{-1}{5}}\right| = \frac{23}{2}$$

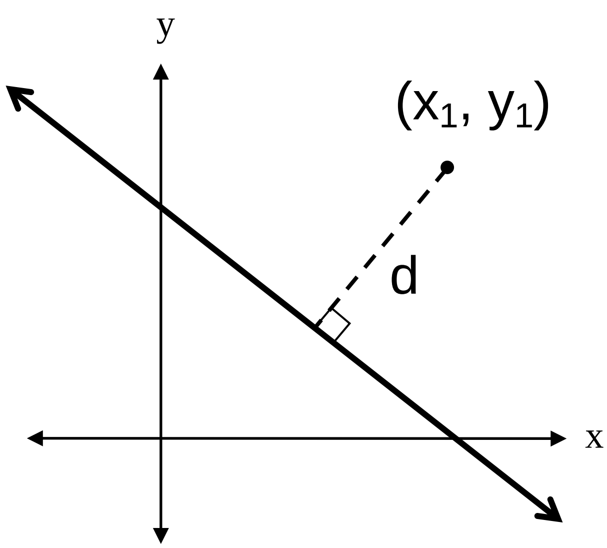
So, if $\tan\theta = \frac{23}{2}$, then

$$\theta = \tan^{-1}\left(\frac{23}{2}\right)$$

$$\theta \approx 1.484 \text{ radians or } 85°$$

The Distance Between a Point and a Line

Finding the distance from a point to a line is finding the length of the perpendicular segment that joins the point to the line.



Distance Between a Point and a Line

The distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Example: Find the distance between the point (2, 3) and the line given by $x - 4y = 1$.

Solution: Put the equation in the form Ax + By + C = 0.

$$x - 4y - 1 = 0$$

Find the distance using the formula:

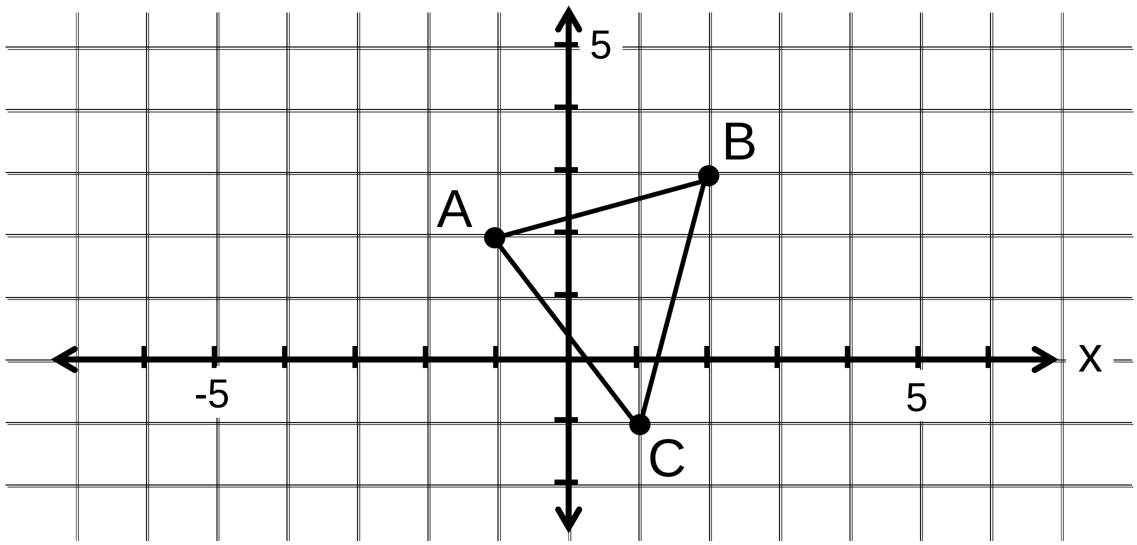
$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|1(2) - 4(3) - 1|}{\sqrt{1^2 + (-4)^2}}$$

$$d = \frac{|2 - 12 - 1|}{\sqrt{1 + 16}}$$

$$d = \frac{11}{\sqrt{17}} = \frac{11\sqrt{17}}{17} \approx 2.67$$

Example: For the triangle with vertices *A*(-1, 2), *B*(2, 3), and *C*(1, -1), find **a)** the altitude, and **b)** the area of the triangle.



a) We will find the distance from *B* to the line *AC*. First find the equation of the line *AC.*

$$m = \frac{2-(-1)}{-1-1} = \frac{3}{-2} = -\frac{3}{2}$$

Use this and one of the points in y = mx + b.

$$y = mx + b$$
$$2 = \frac{-3}{2}(-1) + b$$
$$b = \frac{1}{2}$$

$$y = mx + b$$
$$y = \frac{-3}{2}x + \frac{1}{2}$$
$$2y = -3x + 1$$
$$3x + 2y - 1 = 0$$

Now, using the equation and point B, find the distance from the point B to the line AC. This is the altitude.

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(2) + 2(3) - 1|}{\sqrt{3^2 + 2^2}}$$

$$d = \frac{|6 - 6 - 1|}{\sqrt{9 + 4}}$$

$$d = \frac{1}{\sqrt{13}} = \frac{\sqrt{13}}{13}$$

The altitude is $\frac{\sqrt{13}}{13}$.

b) To find the area, we will need to know the length of AC. To do this, use the points *A*(-1, 2) and *C*(1, -1) in the distance formula.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d = \sqrt{(-1-1)^2 + (2-(-1))^2}$$

$$d = \sqrt{(-2)^2 + (3)^2}$$

$$d = \sqrt{13}$$

So, AC = $\sqrt{13}$.

Now find the area of the triangle.

The formula for the area of a triangle is $A = \frac{1}{2}bh$.

$$A = \frac{1}{2}bh.$$

$$A = \frac{1}{2}(\sqrt{13})\left(\frac{\sqrt{13}}{13}\right)$$

$$A = \frac{1}{2} \text{ square units.}$$