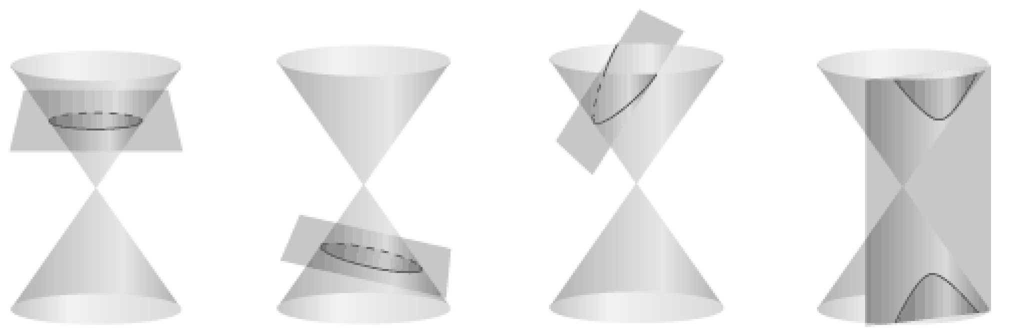
Intro to Conics: Parabolas

Each conic section is the intersection of a plane and a double-napped cone.



Circle Ellipse Parabola Hyperbola

If the plane passes through the vertex of the cone, the resulting figure is a <u>degenerate conic section</u>.

Point Line Two Intersecting Lines

Three Ways to Study Conics

1. Define them in terms of the intersections of planes and cones, as the Greeks did
2. Define them algebraically in terms of the general 2nd degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

3. Define each conic as a locus (collection) of points (x, y) that are equidistant from a fixed point (h, k).

Parabolas

We have previously looked at parabolas as quadratic functions in the form

$$f(x) = ax^2 + bx + c$$

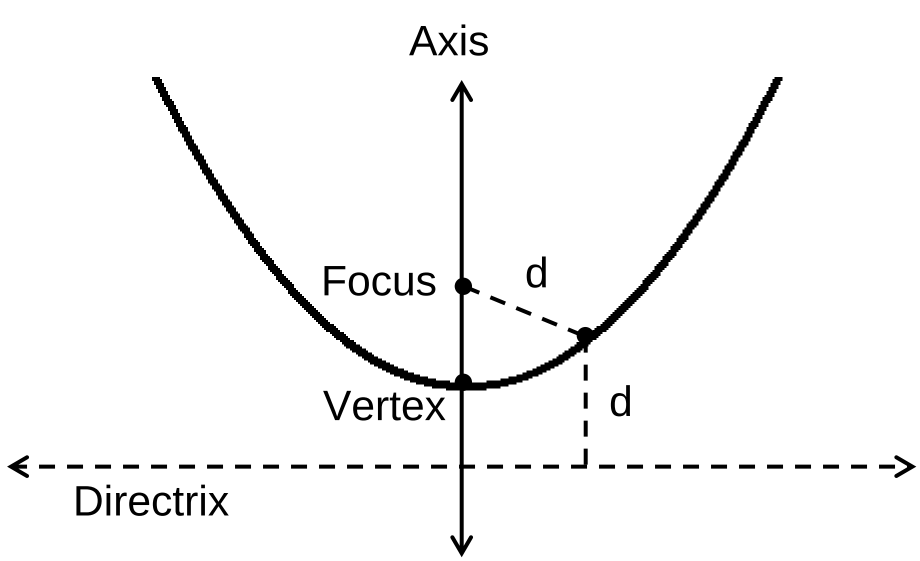
This parabola opens upward if a is positive and downward if a is negative. In addition, there is a vertical stretch or shrink, depending on the absolute value of a. The larger $|a|$ is, the narrower the parabola is.

The following definition is based on the locus of points that make up the parabola, and is independent of the orientation of the parabola.

Definition of Parabola

A <u>parabola</u> as the set of all points (x, y) in a plane that are equidistant from a fixed line and a fixed point not on the line.

- The fixed line is called the <u>directrix</u>.
- The fixed point is called the <u>focus</u>.
- The <u>vertex</u> is the midpoint between the focus and the directrix.
- The <u>axis</u> is the line through the focus and perpendicular to the directrix. (A parabola is always symmetric with respect to its axis.)



Notice that the distance from the focus to the vertex is equal to the distance from the vertex to the directrix.

If the parabola has a directrix parallel to either the x-axis or y-axis, we can derive the following standard equation using the distance formula.

Standard Equation of a Parabola

The standard form of the equation of a parabola with vertex at (h, k) is as follows:

$$(x-h)^2 = 4p(y-k), \qquad p \neq 0$$

Axis: vertical
Directrix: $y = k - p$
Focus: $(h, k + p)$

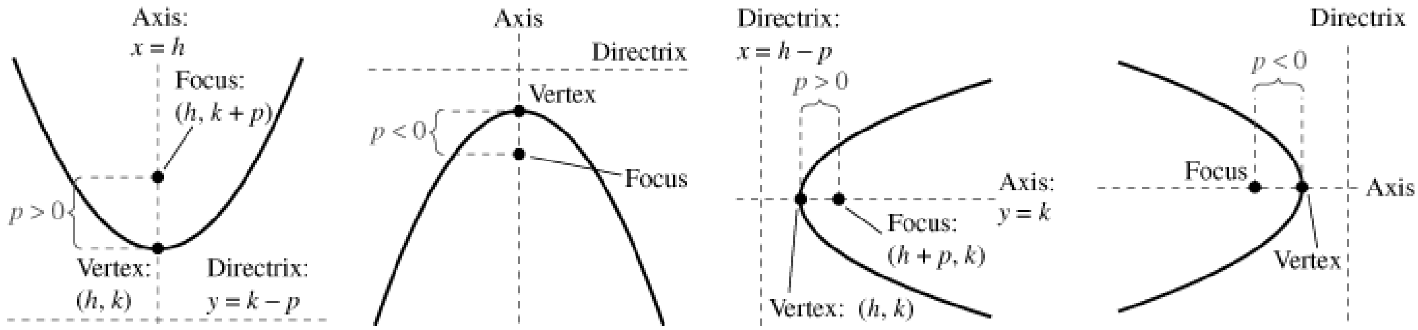
$$(y-k)^2 = 4p(x-h), \qquad p \neq 0$$

Axis: horzontal
Directrix: $x = h - p$
Focus: $(h + p, k)$

The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin (0, 0), the equation takes one of the following forms.

$x^2 = 4py$ vertical axis

$y^2 = 4px$ horzontal axis



(a) $(x-h)^2 = 4p(y-k)$ Vertical axis: $p > 0$

(b) $(x-h)^2 = 4p(y-k)$ Vertical axis: $p < 0$

(c) $(y-k)^2 = 4p(x-h)$ Horizontal axis: $p > 0$

(d) $(y-k)^2 = 4p(x-h)$ Horizontal axis: $p < 0$

Example: Find the standard form of the equation of the parabola with vertex (3,1) and focus (3,4).

Solution: First graph these 2 points to determine the standard equation.

Because the axis is vertical our equation will be of the form

$$(x-h)^2 = 4p(y-k)$$

Fill in the values we know:

$$(x-3)^2 = 4p(y-1)$$

The value of p is the distance from the vertex to the focus, so we have p = 4 - 1 = 3. Substituting gives us

$$(x-3)^2 = 12(y-1)$$

Example: Find the vertex, focus, and directrix of the parabola given by $y = \frac{1}{2}x^2$.

<u>*Solution:*</u> This equation will be of the form $x^2 = 4py$.

If we solve for x^2, we can make the equations match better.

$$y = \frac{1}{2}x^2$$

$$2y = x^2$$

$$x^2 = 2y$$

Looking at $x^2 = 4py$, this means that 4*p* must equal 2.

$$4p = 2$$

$$p = \frac{1}{2}$$

- The vertex is at (0, 0).
- To find the focus, we start at the vertex and go up ½ units to is at (0, ½).
- To find the directrix, we go ½ units down and the equation is y = - ½ .

Example: Find the standard form of the equation of a parabola with vertex at (0, 0) and focus at (-2, 0).

Solution: By sketching these we would see that the directrix is vertical. The directrix would be 2 units to the right of the vertex at x = 2.

Since the focus was 2 units in the negative direction from the vertex, the value of p = -2.

Our standard equation will look like $(y-k)^2 = 4p(x-h)$.
Substituting gives us $(y-0)^2 = 4(-2)(x-0)$.
Simplifying gives us our equation: $y^2 = -8x$.

Note: We could have used the standard equation $y^2 = 4px$ instead of $(y-k)^2 = 4p(x-h)$ because the vertex was at the origin.

Example: Find the vertex, focus, and directrix of the parabola given by

$$x^2 - 2x - 16y - 31 = 0$$

Solution: Before we can find what we are asked to find, we will need to put the equation in standard form by completing the square.

*Get the x terms isolated on one side.

$$x^2 - 2x = 16y + 31$$

*Put in your blanks on both sides and complete the square.

$$x^2 - 2x + \_\_ = 16y + 31 + \_\_$$

$$x^2 - 2x + \underline{1} = 16y + 31 + \underline{1}$$

$$(x-1)^2 = 16y + 32$$

*Factor out the coefficient of y.

$$(x-1)^2 = 16(y+2)$$

- Comparing this to the equation $(x-h)^2 = 4p(y-k)$, we see that the vertex is (1, -2).

We also know that $4p = 16$, which means p = 4.

- For the focus we go 4 units up from the vertex to (1, 2).
- For the directrix we go 4 units down from the vertex to get the line y = -6.