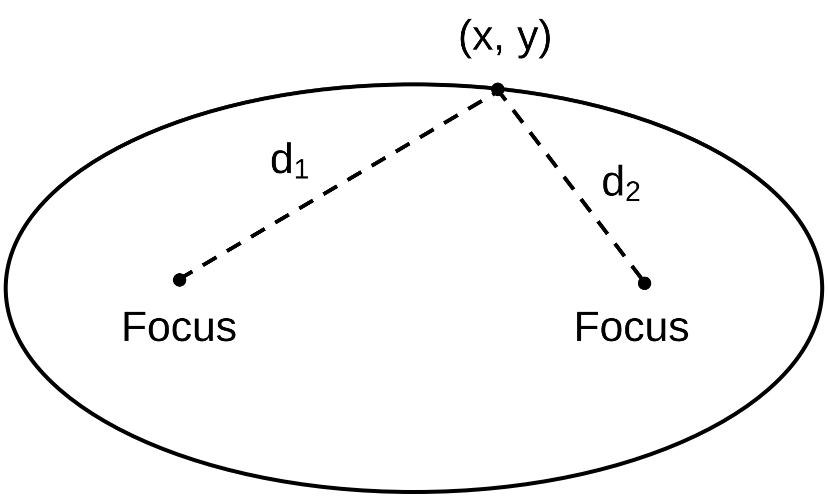
Ellipses

The second type of conic is called an ellipse.

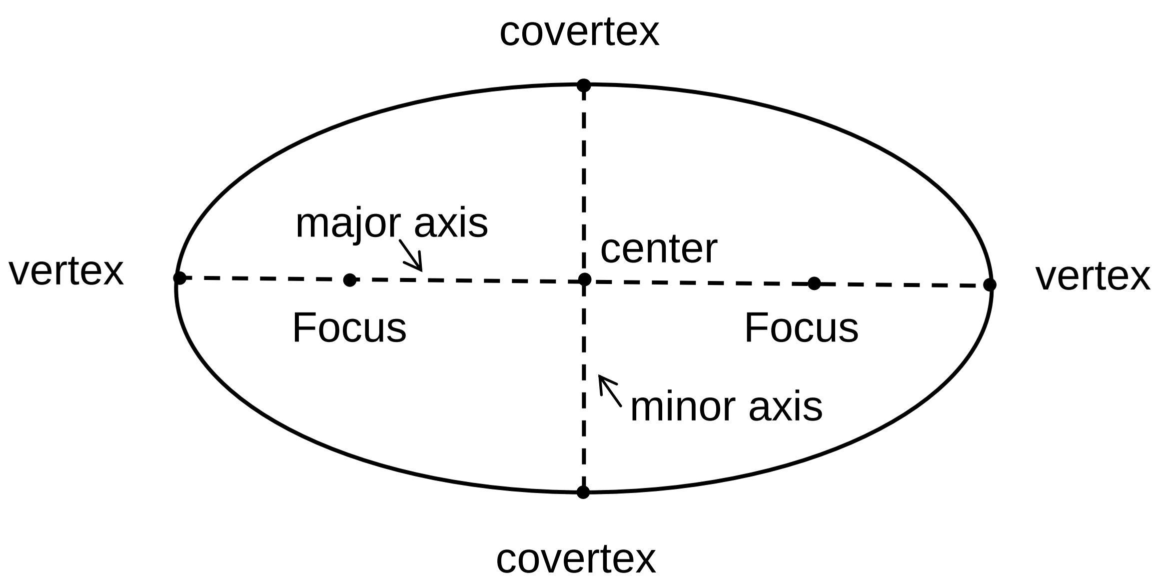
Definition of Ellipse

An ellipse is the set of all points (*x*, *y*) in a plane, the sum of whose distances from two distinct fixed points (**foci**) is constant.



$d_1 + d_2$ is a constant.

There are a number of parts of an ellipse that should be noted:



- The midpoint between the foci is the <u>center</u>.

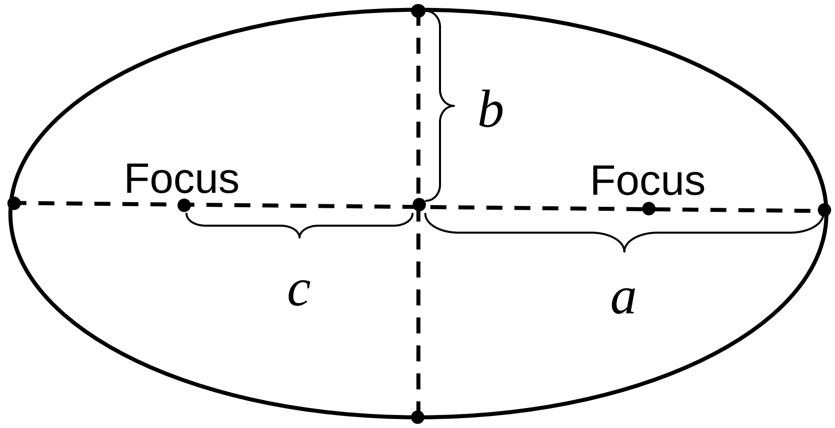
- The line segment through the foci, with endpoints on the ellipses, is the <u>major axis</u>.

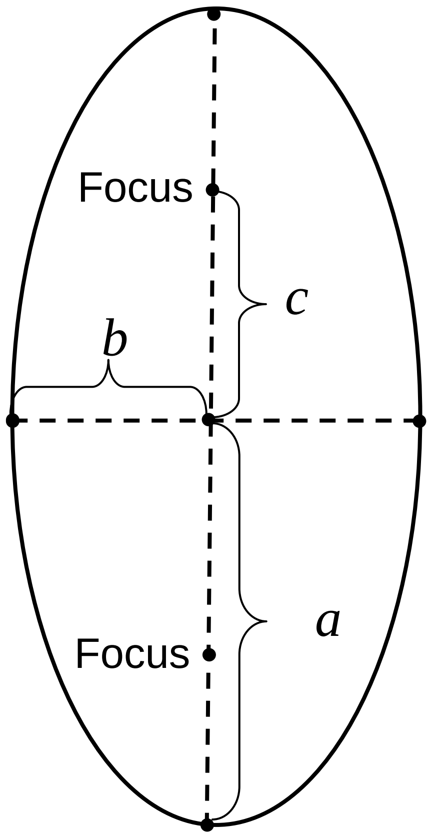
- The endpoints of the major axis are the <u>vertices</u> of the ellipse.

- The line segment through the center and perpendicular to the major axis, with endpoints on the ellipse, is the <u>minor axis</u>.

- The endpoints of the minor axis are the <u>covertices</u> of the ellipse.

There are 3 distances that are important when studying an ellipse.





a is always the longest length
and c is always the focal length

Standard Equation of an Ellipse

The standard form of the equation of an ellipse centered at (h, k), with major axis of length $2a$, minor axis of length $2b$, where 0 < b < a, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$ Major axis is horizontal

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$ Major axis is vertical

The foci lie on the major axis, c units from the center, with $c^2=a^2-b^2$. If the center is at the origin (0, 0), the equation takes one of the following forms.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$ Major axis is horizontal

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$ Major axis is vertical

Example: Find the center, vertices, and foci of the ellipse given by $9x^2 + 4y^2 = 36$.

Solution: First divide through by 36 to get the correct form.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Remembering that the larger number on the bottom corresponds to a, we can see that:

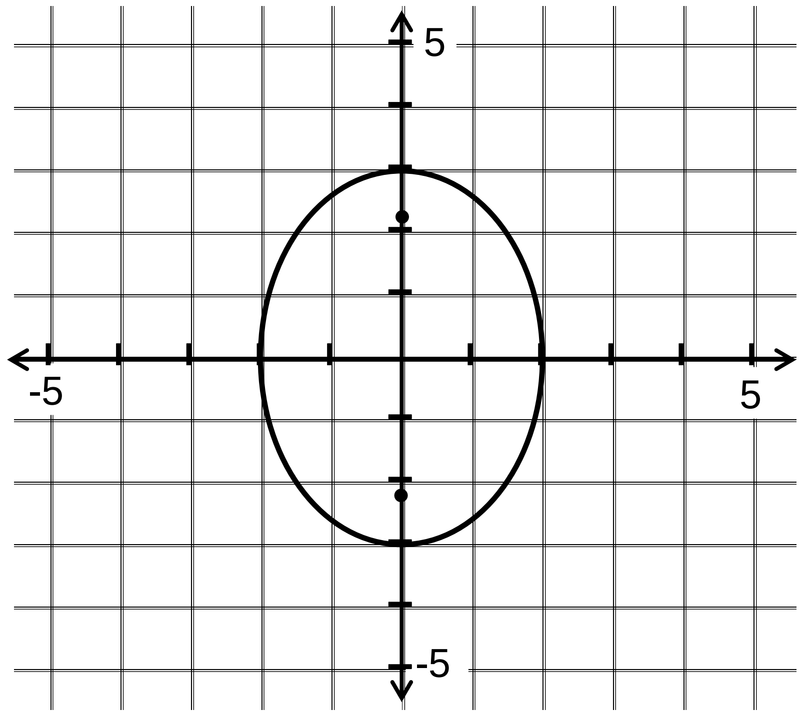
$$b^2 = 4 \quad \text{so} \quad b = 2$$
$$a^2 = 9 \quad \text{so} \quad a = 3$$

Using $c^2 = a^2 - b^2$, we can find c.

$$c^2 = 3^2 - 2^2$$
$$c^2 = 5$$
$$c = \sqrt{5}$$

The center of the ellipse is (0, 0). From that point:

- We go 3 units up and down for the vertices: (0,3), (0,-3)
- We go 2 units right and left for covertices: (2, 0), (-2, 0)
- We go $\sqrt{5}$ units up and down for the foci: $(0, \sqrt{5})$, $(0, -\sqrt{5})$



Example: Find the standard form of the equation of the ellipse centered at the origin with major axis of length 10 and foci at (±3, 0).

Solution: If the major axis is 10, we know that a = 5.
If the foci are at (±3, 0), we know that c = 3.
Solve for b.

$$c^2 = a^2 - b^2$$

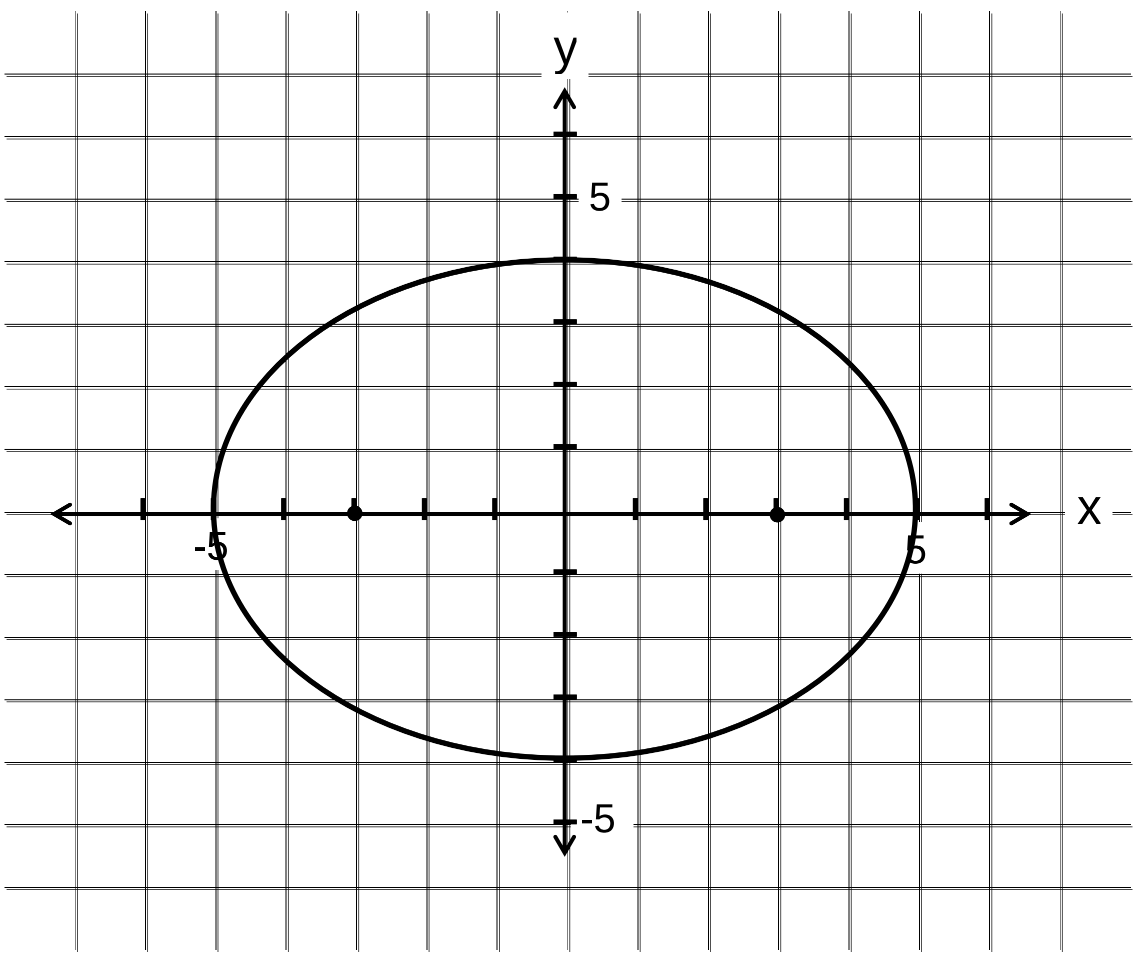
$$3^2 = 5^2 - b^2$$

$$b^2 = 16$$

$$b = 4$$

Since the foci always lie on the major axis, we know that the major axis is horizontal. That tells us that a^2 goes under the x^2 term. Since it is centered at the origin, we have:

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$



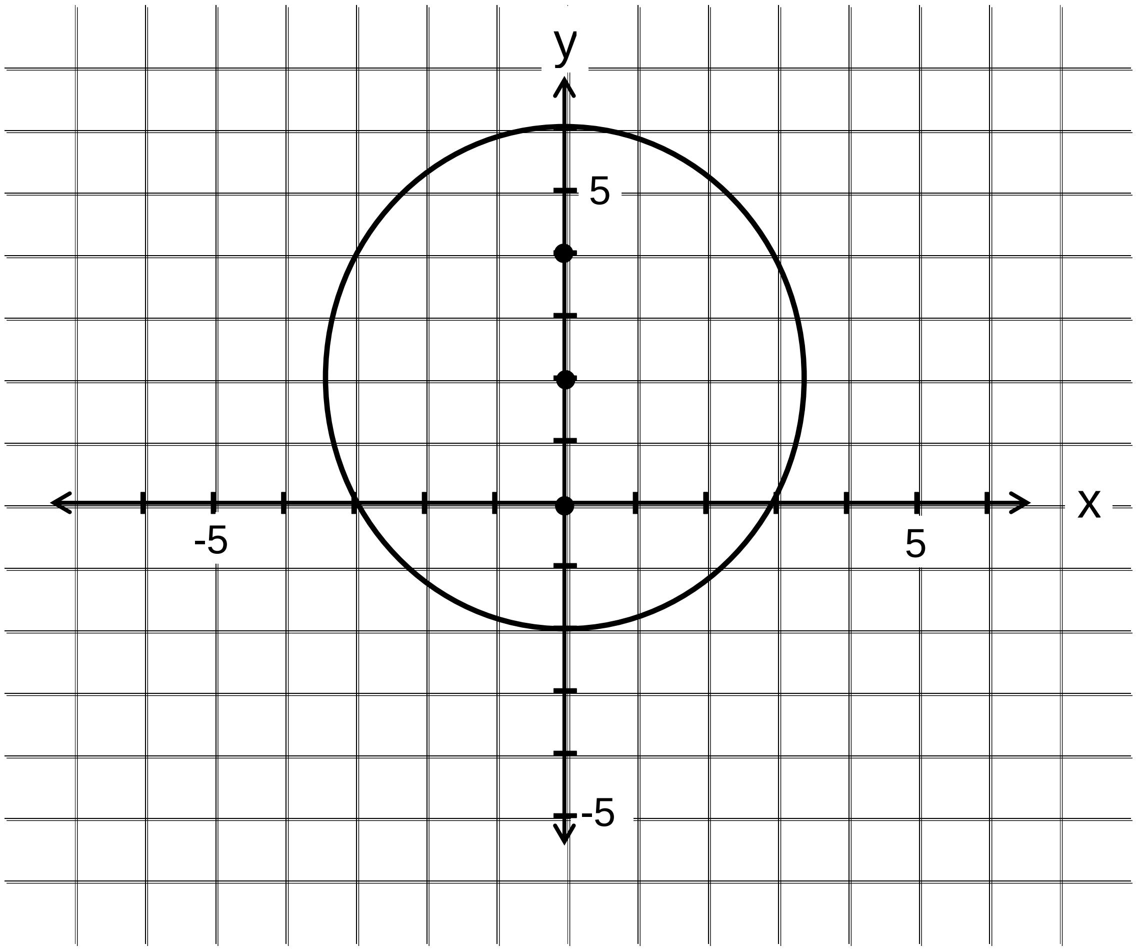
Example: Find the standard form of the equation of the ellipse with foci (0, 0) and (0, 4) with major axis of length of 8.

Solution: If the major axis is 8, we know that a = 4.
If the foci are at (0, 0) and (0, 4), we know that c is half the distance between them so c = 2.
Solve for b.

$$c^2 = a^2 - b^2$$
$$2^2 = 4^2 - b^2$$
$$b^2 = 12$$
$$b = 2\sqrt{3}$$

Since the foci always lie on the major axis, we know that the major axis is vertical. That tells us that a^2 goes under the y^2 term. The center is half way between the foci, so the center must be (0, 2). Thus we have

$$\frac{(x-0)^2}{12} + \frac{(y-2)^2}{16} = 1$$



Example: Sketch the graph of $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

<u>Solution</u>: You need to complete the square with the x-terms and the y-terms.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0$$

*Get the constant on the other side and group the x-terms together and the y-terms terms, putting in blanks on both sides of the equation.

$$(x^2 + 6x + \_\_) + (4y^2 - 8y + \_\_) = -9 + \_\_ + \_\_$$

*Before completing the square, pull the 4 out of the y group.

$$(x^2 + 6x + \_\_) + 4(y^2 - 2y + \_\_) = -9 + \_\_ + \_\_$$

*Complete the squares.

$$(x^2 + 6x + \underline{9}) + 4(y^2 - 2y + \underline{1}) = -9 + \underline{9} + \underline{4}$$

*Remember to add 4(1) on the right for the y group.

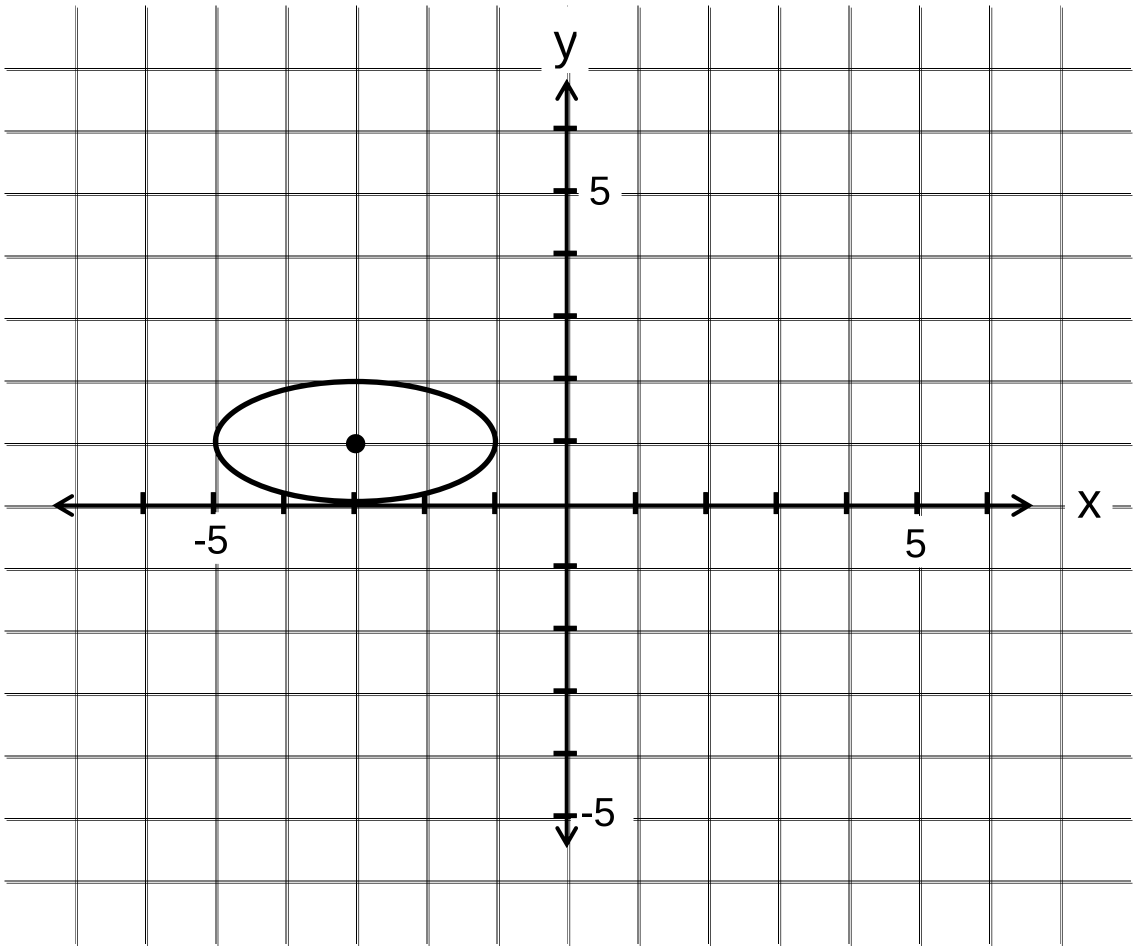
$$(x+3)^2 + 4(y-1)^2 = 4$$

*Because we need a 1 on the right, divide through by 4.

$$\frac{(x+3)^2}{4} + \frac{4(y-1)^2}{4} = \frac{4}{4}$$

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1$$

<u>center</u>: (-3, 1)
<u>major axis</u>: horizontal and $a = 2$
<u>minor axis</u>: vertical and $b = 1$



Example: Find the center, vertices, and foci of the ellipse

$$16x^2 + 9y^2 - 96x + 36y + 36 = 0$$

Solution: Put the equation in standard form by completing the squares.

$$16x^2 + 9y^2 - 96x + 36y + 36 = 0$$

$$(16x^2 - 96x + \_\_) + (9y^2 + 36y + \_\_) = -36 + \_\_ + \_\_$$

$$16(x^2 - 6x + \_\_) + 9(y^2 + 4y + \_\_) = -36 + \_\_ + \_\_$$

$$16(x^2 - 6x + \underline{9}) + 9(y^2 + 4y + \underline{4}) = -36 + \underline{144} + \underline{36}$$

$$16(x-3)^2 + 9(y+2)^2 = 144$$

$$\frac{16(x-3)^2}{144} + \frac{9(y+2)^2}{144} = \frac{144}{144}$$

$$\frac{(x-3)^2}{9} + \frac{(y+2)^2}{16} = 1$$

center: (3, -2)
major axis: vertical and a = 4
minor axis: horizontal and b = 3

Find the foci by first finding c.

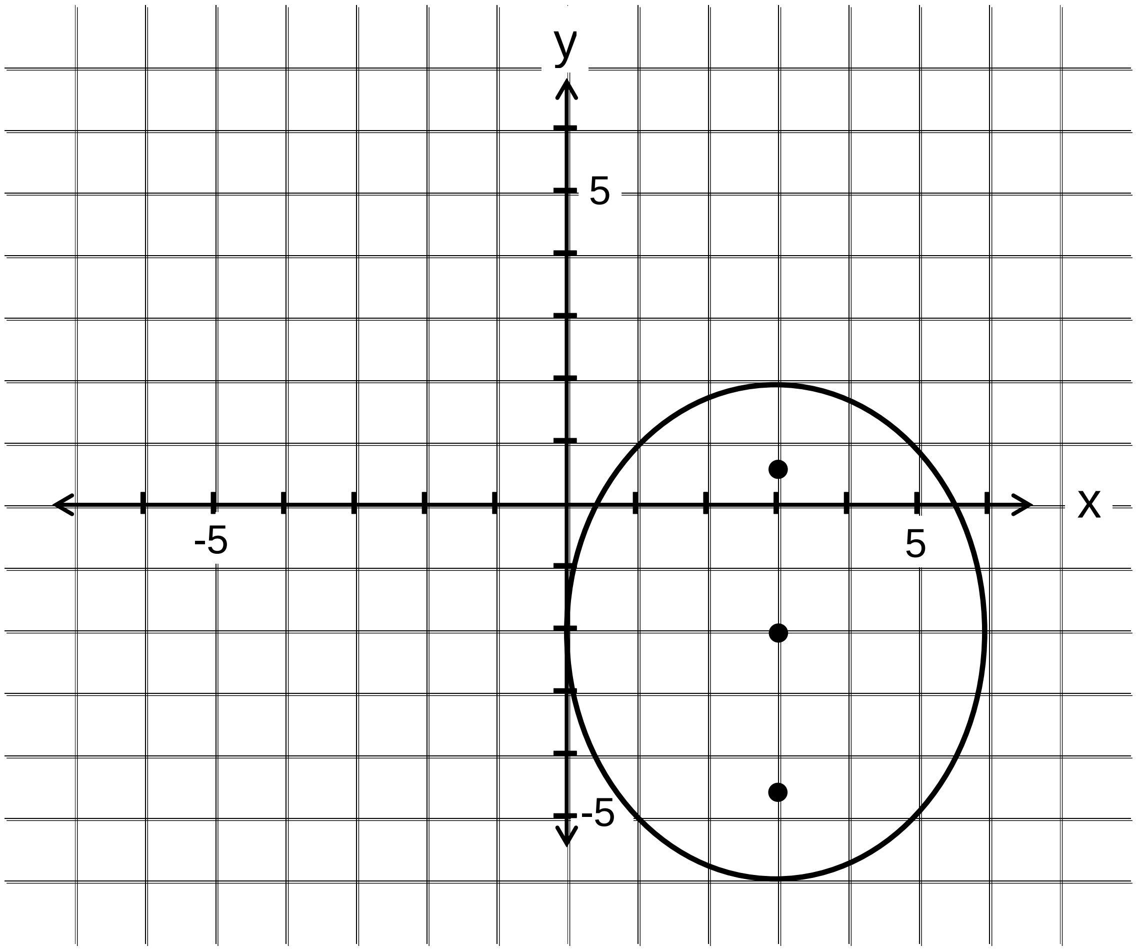
$$c^2 = a^2 - b^2$$

$$c^2 = 4^2 - 3^2$$

$$c^2 = 7$$

$$c = \sqrt{7}$$

The foci are $\sqrt{7}$ units above and below the center, so the coordinates would be (3, -2+$\sqrt{7}$) and (3, -2-$\sqrt{7}$).



Eccentricity

To measure the ovalness of an ellipse, we use the concept of eccentricity.

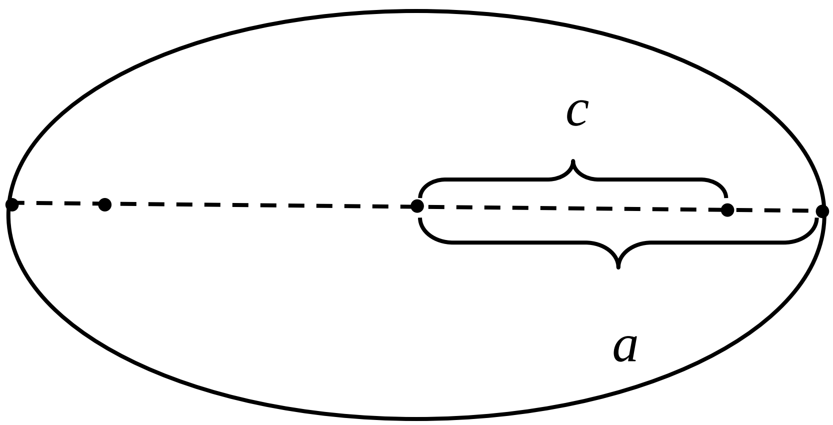
Definition of Eccentricity

The eccentricity e of an ellipse is given by the ratio

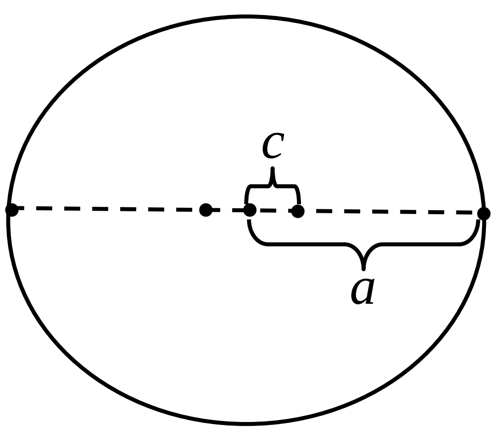
$$e = \frac{c}{a}$$

*<u>Note</u>: Because a is always greater than c, the fraction $\frac{c}{a}$ will always be between 0 and 1.

- When c is close to a, that means the foci are close the the vertices and the fraction $\frac{c}{a}$ is close to 1.



- When c is much smaller than a, that means the foci are close to the center and the fraction $\frac{c}{a}$ is small.



Example: Find the eccentricity of the ellipse

$$\frac{(x+5)^2}{9}+\frac{(y-1)^2}{25}=1$$

Solution: Find c.

$$c^2=a^2-b^2$$
$$c^2=5^2-3^2$$
$$c^2=16$$
$$c=4$$

The eccentricity is $e=\frac{c}{a}=\frac{4}{5}$

Example: Find the eccentricity of the ellipse

$$\frac{(x-1)^2}{5}+\frac{(y+7)^2}{4}=1$$

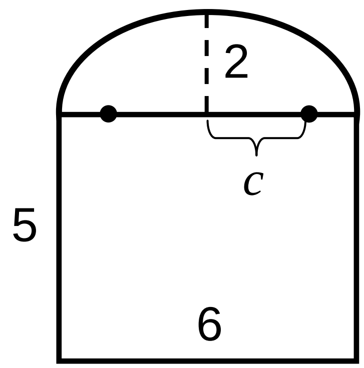
$$c^2=a^2-b^2$$
$$c^2=5-4$$
$$c^2=1$$
$$c=1$$

so

$$e=\frac{c}{a}$$
$$e=\frac{1}{\sqrt{5}}$$
$$e=\frac{\sqrt{5}}{5}$$

Application

Example: A passageway in a house is to have straight sides and a semielliptically-arched top. The straight sides are 5 feet tall and the passageway is 7 feet tall at its center and 6 feet wide. Where should the foci be located to make the template for the arch?



a = 3 and b = 2. Find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 3^2 - 2^2$$
$$c^2 = 5$$
$$c = \sqrt{5} \approx 2.236$$

The foci should be placed 2.236 feet to the right and left of the center of the semiellipse.