Rotation of Conics

We know that equations of conics with axes parallel to one of the coordinate axes can be written as in the general form of $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

General Equations of Conics

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following:

1. A circle if $A = C$
2. A parabola if $AC = 0$ *(A=0 or C=0,but not both)*
3. An ellipse if $AC > 0$ *(A and C have like signs)*
4. A hyperbola if $AC < 0$ *(A and C have unlike signs)*

If the conic has axes that are rotated so they are not parallel to either the x-axis or y-axis, that conic has the general equation of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

**The difference between this equation and the one above is that this equation includes an xy-term. We cannot complete the square because of this term.

To eliminate this xy-term we will use a procedure called rotation of axes. We will rotate the x- and y-axes until they are parallel to the axes of the conic. We will call the new axes the x'-axis and y'-axis. Then conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

Since this equation has no xy-term, we can complete the square to get a standard form of one of the conics.

Rotation of Axes to Eliminate an xy-Term

The general 2nd-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

The coefficients of the new equation are obtained by making the substitutions

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta$$

Example: Write the equation $xy-1=0$ in standard form.

<u>*Solution*</u>:

1. Determine the values of A, B, and C.

   - *Since there is not A or C in the above equation, so A=0 and C=0.*
   - *The coefficient of xy is 1, so B=1.*

2. Put A, B, and C into the angle equation and solve for *θ*.

$$\cot 2\theta = \frac{A-C}{B}$$

$$\cot 2\theta = \frac{0-0}{1}$$

$$\cot 2\theta = 0$$

The cotangent = 0 when the angle is $\frac{\pi}{2}$*.*

Thus, $2\theta = \frac{\pi}{2}$*.*

Solving this gives us $\theta = \frac{\pi}{4}$*.*

3. Solve for x and y using the value of θ.

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} \qquad y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4}$$

$$x = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) \qquad y = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right)$$

$$x = \frac{x'\sqrt{2} - y'\sqrt{2}}{2} \qquad y = \frac{x'\sqrt{2} + y'\sqrt{2}}{2}$$

4. Substitute these values in for x and y in the original equation.

$$xy - 1 = 0$$

$$\left(\frac{x'\sqrt{2} - y'\sqrt{2}}{2}\right)\left(\frac{x'\sqrt{2} + y'\sqrt{2}}{2}\right) - 1 = 0$$

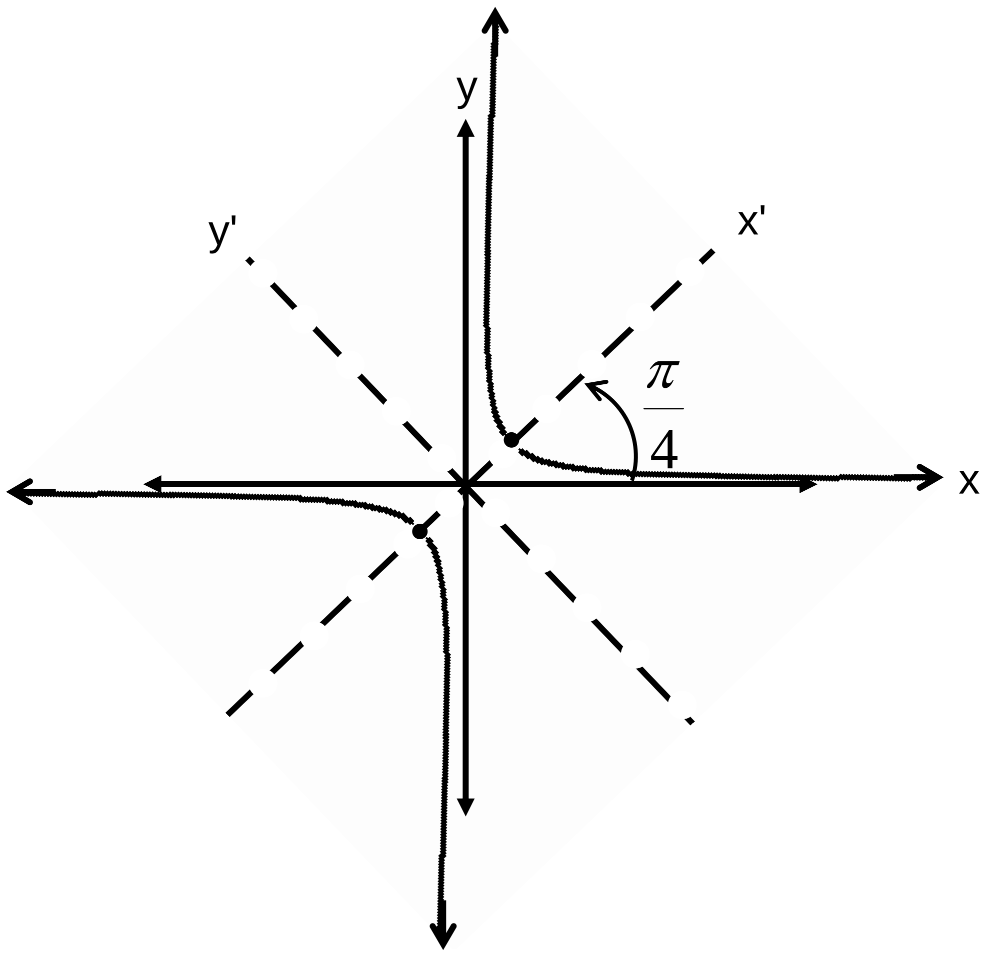
$$\left(\frac{x'\sqrt{2} - y'\sqrt{2}}{2}\right)\left(\frac{x'\sqrt{2} + y'\sqrt{2}}{2}\right) = 1$$

$$\frac{2(x')^2 - 2(y')^2}{4} = 1$$

$$\frac{(x')^2}{2} - \frac{(y')^2}{2} = 1$$

5. Graph the equation.

- Draw the axes rotated θ degrees (i.e. $\frac{\pi}{4}$).
- This is a hyperbola centered at the origin with vertices $(\sqrt{2},0)$ and $(-\sqrt{2},0)$ on the $x'y'$-plane.



6. To find the vertices in the xy-plane, substitute the coordinates from the $x'y'$-plane into the equations for x and y.

$(\sqrt{2},0)$ in the $x'y'$-plane

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$x = (\sqrt{2})\cos\frac{\pi}{4} - (0)\sin\frac{\pi}{4} \qquad y = (\sqrt{2})\sin\frac{\pi}{4} + (0)\cos\frac{\pi}{4}$$

$$x = (\sqrt{2})\left(\frac{\sqrt{2}}{2}\right) - 0 \qquad y = (\sqrt{2})\left(\frac{\sqrt{2}}{2}\right) + 0$$

$$x = \frac{2}{2} = 1 \qquad y = \frac{2}{2} = 1$$

- The point $(\sqrt{2},0)$ in the $x'y'$-plane coincides with (1,1) in the xy-plane.
- In the same way, $(-\sqrt{2},0)$ in the $x'y'$-plane coincides with (-1,-1) in the xy-plane.

Example: Sketch the graph of $x^2 + \sqrt{3}xy + 2y^2 - 2 = 0$.

<u>*Solution*</u>:

1. Determine the values of A, B, and C.

   - *A=1, B=*$\sqrt{3}$*, and C=2.*

2. Put A, B, and C into the angle equation and solve for *θ*.

$$\cot 2\theta = \frac{A - C}{B}$$

$$\cot 2\theta = \frac{1 - 2}{\sqrt{3}}$$

$$\cot 2\theta = \frac{-1}{\sqrt{3}}$$

The cotangent = $\frac{-1}{\sqrt{3}}$ *when the angle is* $\frac{2\pi}{3}$*.*

Thus, 2θ = $\frac{2\pi}{3}$*.*

Solving this gives us θ = $\frac{\pi}{3}$*.*

3. Solve for x and y using the value of θ.

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$x = x'\cos\frac{\pi}{3} - y'\sin\frac{\pi}{3} \qquad y = x'\sin\frac{\pi}{3} + y'\cos\frac{\pi}{3}$$

$$x = x'\left(\frac{1}{2}\right) - y'\left(\frac{\sqrt{3}}{2}\right) \qquad y = x'\left(\frac{\sqrt{3}}{2}\right) + y'\left(\frac{1}{2}\right)$$

$$x = \frac{x' - y'\sqrt{3}}{2} \qquad y = \frac{x'\sqrt{3} + y'}{2}$$

4. Substitute these values in for x and y in the original equation.

$$x^2 + \sqrt{3}xy + 2y^2 - 2 = 0$$

$$\left(\frac{x' - y'\sqrt{3}}{2}\right)^2 + \sqrt{3}\left(\frac{x' - y'\sqrt{3}}{2}\right)\left(\frac{x'\sqrt{3} + y'}{2}\right) + 2\left(\frac{x'\sqrt{3} + y'}{2}\right)^2 - 2 = 0$$

$$\frac{x'^2 - 2x'y'\sqrt{3} + 3y'^2}{4} + \frac{3x'^2 + \sqrt{3}x'y' - 3\sqrt{3}x'y' - 3y'^2}{4} + \frac{6x'^2 + 4x'y'\sqrt{3} + 2y'^2}{4} = 2$$

This simplifies to:

$$\frac{10x'^2 + 2y'^2}{4} = 2$$

$$10x'^2 + 2y'^2 = 8$$

$$\frac{10x'^2}{8} + \frac{2y'^2}{8} = \frac{8}{8}$$

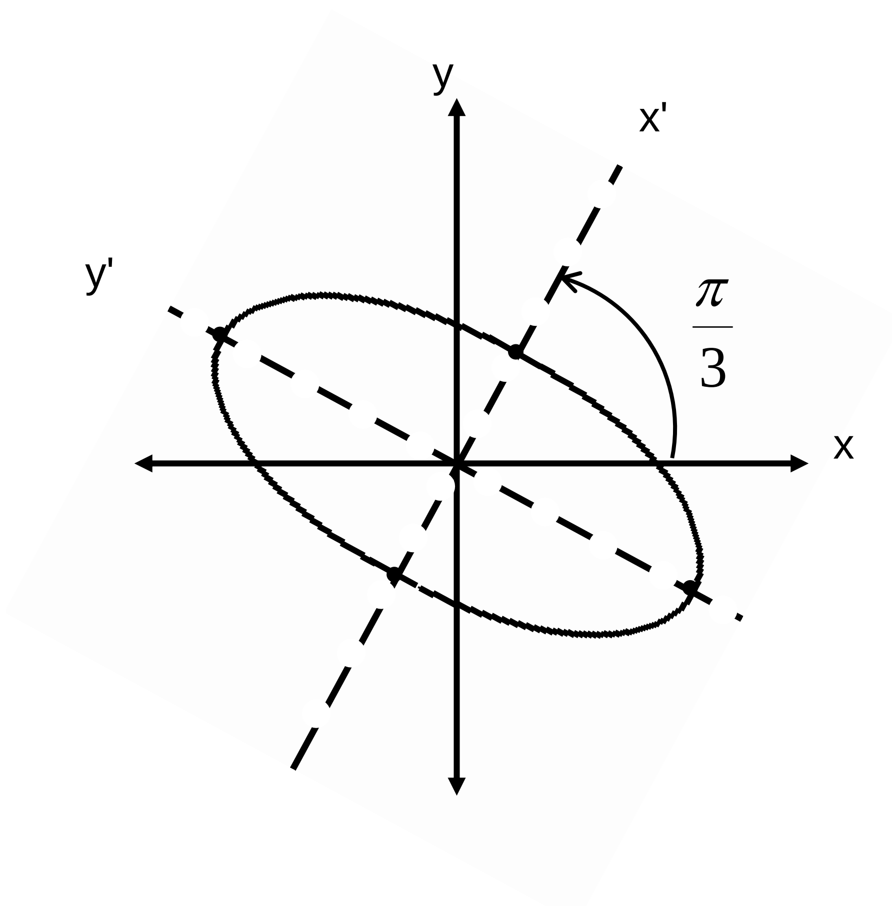
$$\frac{5x'^2}{4} + \frac{y'^2}{4} = 1$$

$$\frac{\left(\frac{1}{5}\right)5x'^2}{\left(\frac{1}{5}\right)4} + \frac{y'^2}{4} = 1$$

$$\frac{x'^2}{\left(\frac{4}{5}\right)} + \frac{y'^2}{4} = 1$$

5. Graph the equation.

- Draw the axes rotated θ degrees (i.e. $\frac{\pi}{3}$).
- This is an ellipse centered at the origin with a vertical major axis where a = 2 and b = $\frac{2}{\sqrt{5}}$. 0.9. The vertices are (0, "2) and the covertices are (±0.9, 0) in the $x'y'$-plane.



6. To find the vertices in the *xy*-plane, substitute the coordinates from the $x'y'$-plane into the equations for x and y.

$(0,2)$ in the $x'y'$-plane

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$x = (0)\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3} \qquad y = 0\sin\frac{\pi}{3} + 2\cos\frac{\pi}{3}$$

$$x = 0 - 2\left(\frac{\sqrt{3}}{2}\right) \qquad y = 0 + 2\left(\frac{1}{2}\right)$$

$$x = -\sqrt{3} \qquad y = 1$$

- The point (0, 2) in the $x'y'$-plane coincides with $(-\sqrt{3},1)$ in the *xy*-plane.

- In the same way, (0, -2) in the $x'y'$-plane coincides with $(-\sqrt{3}, -1)$ in the *xy*-plane.

Example: Sketch the graph of

$$3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$$

<u>*Solution*</u>:

1. Determine the values of A, B, and C.

   - A=3, B=2$\sqrt{3}$, and C=1.

2. Put A, B, and C into the angle equation and solve for *θ*.

$$\cot 2\theta = \frac{A-C}{B}$$

$$\cot 2\theta = \frac{3-1}{2\sqrt{3}}$$

$$\cot 2\theta = \frac{2}{2\sqrt{3}}$$

$$\cot 2\theta = \frac{1}{\sqrt{3}}$$

The cotangent = $\frac{1}{\sqrt{3}}$ *when the angle is* $\frac{\pi}{3}$.

Thus, 2θ = $\frac{\pi}{3}$ *and θ =* $\frac{\pi}{6}$.

3. Solve for x and y using the value of θ.

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} \qquad y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6}$$

$$x = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) \qquad y = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right)$$

$$x = \frac{x'\sqrt{3} - y'}{2} \qquad y = \frac{x' + y'\sqrt{3}}{2}$$

4. Substitute these values in for x and y in the original equation.

$$3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$$

$$3\left(\frac{x'\sqrt{3} - y'}{2}\right)^2 + 2\sqrt{3}\left(\frac{x'\sqrt{3} - y'}{2}\right)\left(\frac{x' + y'\sqrt{3}}{2}\right) + \left(\frac{x' - y'\sqrt{3}}{2}\right)^2 + 2\left(\frac{x'\sqrt{3} - y'}{2}\right) - 2\sqrt{3}\left(\frac{x' - y'\sqrt{3}}{2}\right) = 0$$

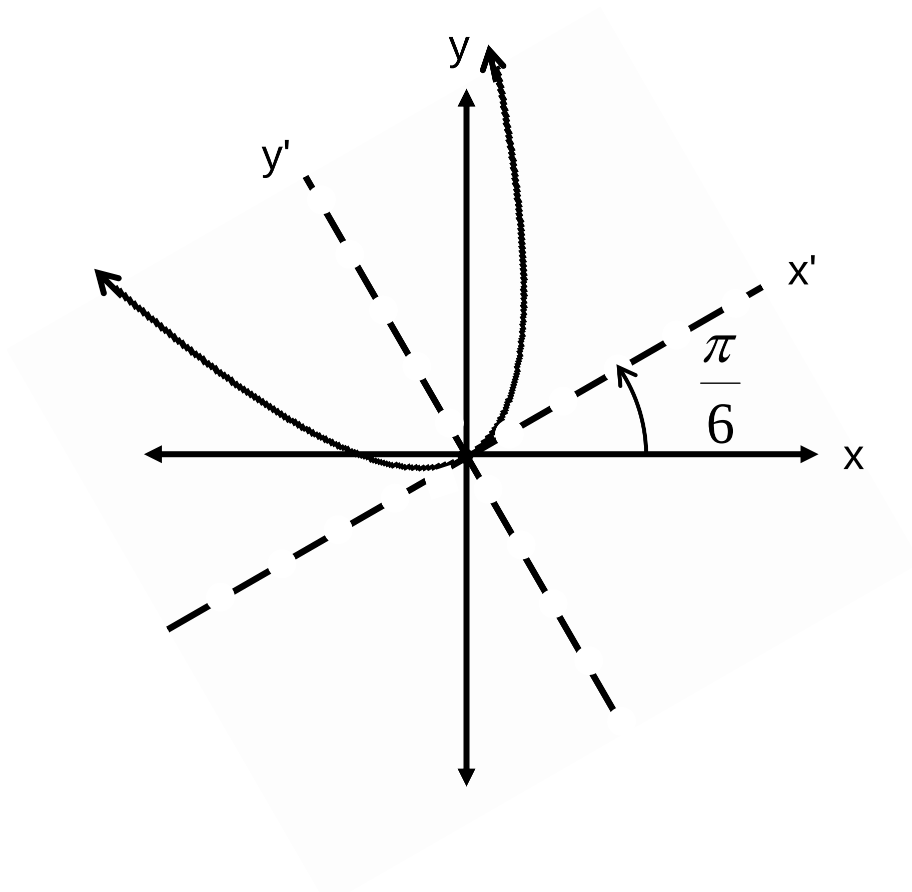
$$4x'^2 - 4y' = 0$$

$$4y' = 4x'^2$$

$$y' = x'^2$$

5. Graph the equation.

- Draw the axes rotated θ degrees (i.e. $\frac{\pi}{6}$).
- This is a parabola centered at the origin, opening up, with the standard shape.



6. To find the vertices in the xy-plane, substitute the coordinates from the $x'y'$-plane into the equations for x and y.

(0, 0) in the $x'y'$-plane

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$x = (0)\cos\frac{\pi}{6} - 0\sin\frac{\pi}{6} \qquad y = 0\sin\frac{\pi}{6} + 0\cos\frac{\pi}{6}$$

$$x = 0 \qquad y = 0$$

- The point (0, 0) in the $x'y'$-plane coincides with (0, 0) in the xy-plane.