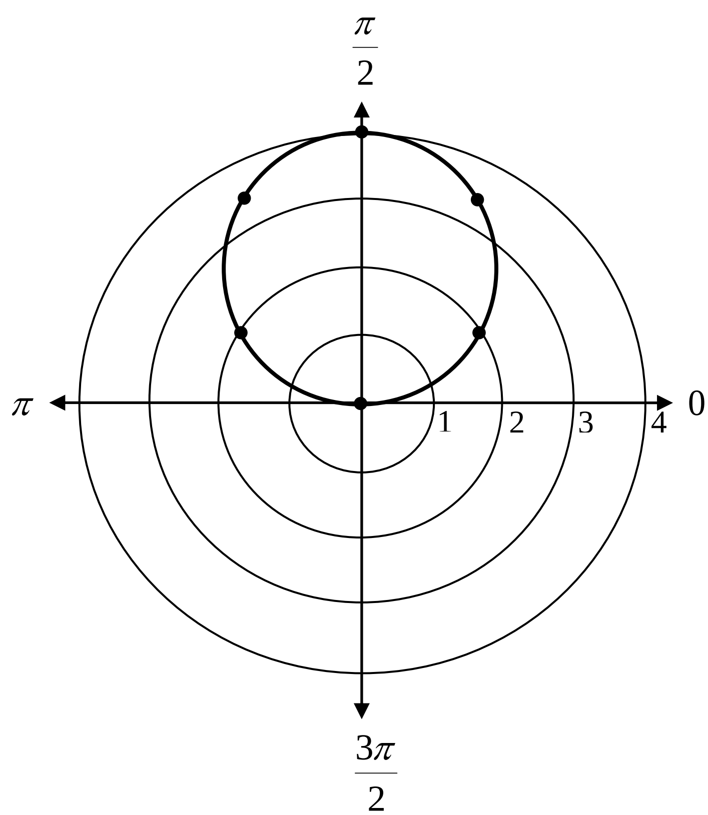
Graphs of Polar Equations

To begin graphing in the polar coordinate system we will start with plotting points.

Look at the polar equation $r = 4\sin\theta$. Make a table.

θ	r
0	0
$\frac{\pi}{6}$	2
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.5$
$\frac{\pi}{2}$	4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.5$
$\frac{5\pi}{6}$	2
π	0
$\frac{7\pi}{6}$	-2
$\frac{3\pi}{2}$	-4
$\frac{11\pi}{6}$	-2
2π	0

Graph the points and then connect them.



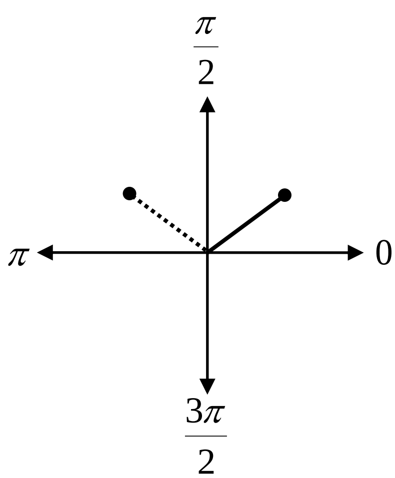
Notice that the points from π to 2π retrace what is already graphed.

Symmetry

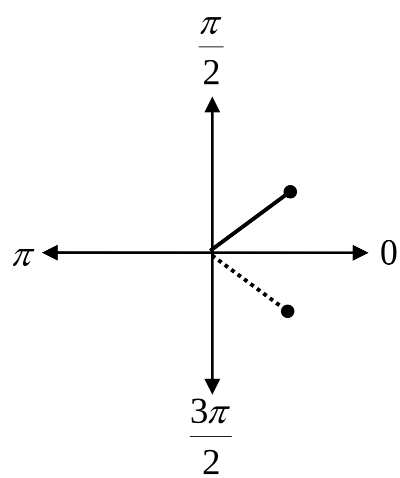
Just as symmetry helps us to graph equations in rectangular form, it also helps us to graph in polar form.

The graph above shows symmetry with respect to the y-axis. But in polar coordinates the y-axis is the line $\theta = \frac{\pi}{2}$.

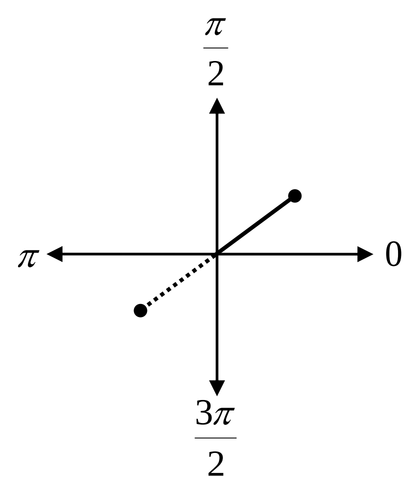
In general, we have 3 types of symmetry for polar graphs.



Symmetry with respect to the line $\theta = \frac{\pi}{2}$



Symmetry with respect to the polar axis



Symmetry with respect to the pole.

Tests for Symmetry in Polar Coordinates

The graph of a polar equation is symmetric with respect to the following if the given substitution yields an equivalent equation.

1. *The line* $\theta = \frac{\pi}{2}$: Replace (r, θ) with $(r, \pi - \theta)$
   or $(-r, -\theta)$
2. *The polar axis*: Replace (r, θ) with $(r, -\theta)$
   or $(-r, \pi - \theta)$
3. *The pole*: Replace (r, θ) with $(r, \pi + \theta)$
   or $(-r, \theta)$

You will have to refer back to the sum and difference formulas from chapter 5:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

You may also need to refer back to the Odd/Even identities of chapter 4:

$$\cos(-\theta) = \cos(\theta) \qquad \sec(-\theta) = \sec(\theta)$$
$$\sin(-\theta) = -\sin(\theta) \qquad \csc(-\theta) = -\csc(\theta)$$
$$\tan(-\theta) = -\tan(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

Example: Describe the symmetry of the polar equation $r = 2(1 - \sin\theta)$.

Solution: Test for each type of symmetry.

1. *The line* $\theta = \dfrac{\pi}{2}$: Replace (r, θ) with $(r, \pi - \theta)$ or $(-r, -\theta)$. Let's pick $(r, \pi - \theta)$.

$$r = 2(1 - \sin\theta)$$
$$r = 2(1 - \sin(\pi - \theta))$$

Use the sum and difference formula for sine to see if $\sin\theta = \sin(\pi - \theta)$.

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\sin(\pi-\theta) = \sin \pi \cos \theta - \cos \pi \sin \theta$$
$$\sin(\pi-\theta) = (0)\cos \theta - (-1)\sin \theta$$
$$\sin(\pi-\theta) = \sin \theta$$

So, $r = 2(1-\sin(\pi-\theta)) = 2(1-\sin\theta)$.

*Note: To see if $\sin\theta = \sin(\pi-\theta)$, you can think through this intuitively as well. Give yourself an example. Does $\sin\frac{\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right)$?

$$\sin\frac{\pi}{6} = \frac{1}{2}$$
$$\sin\left(\pi-\frac{\pi}{6}\right) = \sin\frac{5\pi}{6} = \frac{1}{2}$$

The answer is yes. They are the same.

The graph *is* symmetric to the line $\theta = \frac{\pi}{2}$. We don't have to check $(r, \pi-\theta)$.

2. *The polar axis*: Replace (r,θ) with $(r,-\theta)$ or $(-r,\pi-\theta)$. Let's pick $(r,-\theta)$.

$$r = 2(1-\sin\theta)$$
$$r = 2(1-\sin(-\theta))$$

Refering to the odd/even identities above, we see that $\sin(-\theta) = -\sin\theta$. Therefore, $\sin(-\theta) \neq \sin\theta$, we do <u>*not*</u> get an equivalent equation.

*<u>Note</u>: We can think through this intuitively also.

Does $\sin\theta = \sin(-\theta)$?

For example, does $\sin\frac{\pi}{4} = \sin\frac{-\pi}{4}$?

The answer is <u>no</u>. Therefore, replacing (r,θ) with $(r,-\theta)$ does <u>*not*</u> give us an equivalent equation.

**The polar axis*: What if we chose $(-r,\pi-\theta)$ instead?

$$r = 2(1-\sin\theta)$$
$$-r = 2(1-\sin(\pi-\theta))$$

We know from above, that $\sin\theta = \sin(\pi-\theta)$, so we get

$$-r = 2(1-\sin\theta)$$

This is <u>*not*</u> an equivalent equation.

3. *The pole*: Replace (r,θ) with $(r,\pi+\theta)$ or $(-r,\theta)$.

Let's choose $(-r,\theta)$.

$$r = 2(1-\sin\theta)$$
$$-r = 2(1-\sin\theta)$$

This does ***not*** give us an equivalent equation either.

*Note: Polar graph can exhibit symmetry even when the tests *fail* to indicate symmetry. The test will show if symmetry does exists, but it can not conclusively prove that symmetry *does not* exist.

Quick Tests for symmetry:

1. The graph of $r = f(\sin\theta)$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

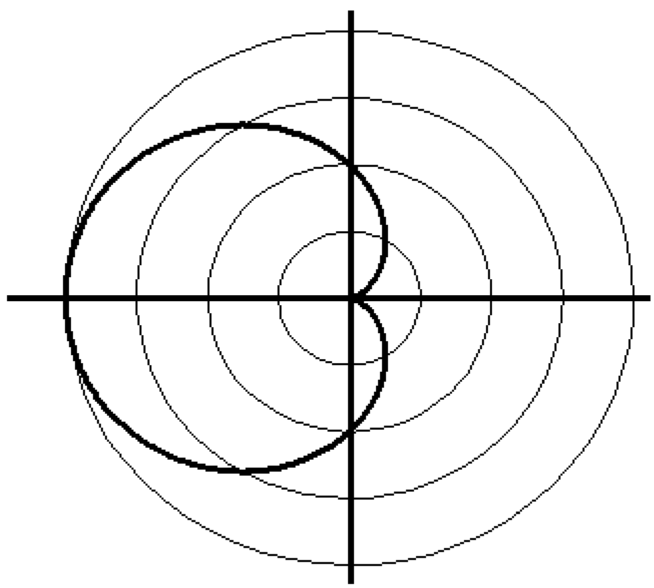
2. The graph of $r = g(\cos\theta)$ is symmetric with respect to the polar axis.

*You may use the quick tests to find one symmetry, but you must use the tests to determine the other 2 symmetries.

Example: Use symmetry to graph the equation
$r = 2 - 2\cos\theta$.

Solution: Because this equation is a function of cosine, we know there is symmetry with respect to the polar axis. Make a table and draw the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	2π
r	0	≈ 0.3	≈ 0.6	2	3	≈ 3.4	4	0



$$y = 2 - 2\cos\theta$$

This graph is called a limaçon.

Zeros and Maximum *r*-Values

Two other values will help us in our graphing:

- points at which |*r*| is maximum
- points at which *r* = 0.

Example: Consider the equation $r = 1 - 2\cos\theta$.

➢ Where will |*r*| be at a maximum?

- $\cos\theta$ is the largest at $\theta = 0$ $(\cos 0 = 1)$.
- $\cos\theta$ is the smallest at $\theta = \pi$ $(\cos\pi = -1)$.

When $\cos 0 = 1$, then

$$\begin{aligned} r &= 1 - 2\cos\theta \\ &= 1 - 2\cos 0 \\ &= 1 - 2(1) \\ &= -1 \\ |r| &= 1 \end{aligned}$$

When $\cos\pi = -1$, then

$$\begin{aligned} r &= 1 - 2\cos\theta \\ &= 1 - 2\cos\pi \\ &= 1 - 2(-1) \\ &= 3 \\ |r| &= 3 \end{aligned}$$

Therefore, the max for |*r*| happens when $\theta = \pi$.

- What are the zeros of the equation?

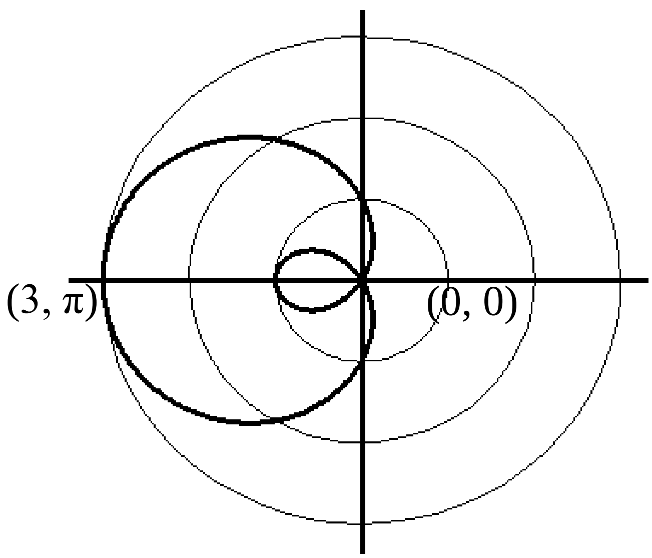
To find the zeros, let r = 0 and solve.

$$r = 1 - 2\cos\theta \qquad \cos\theta = \frac{1}{2},$$

$$0 = 1 - 2\cos\theta$$

$$2\cos\theta = 1 \qquad \theta = \frac{\pi}{3}, \frac{5\pi}{3}, \text{etc.}$$

Since the equation is a function of cosine, we know we have symmetry with respect to the polar axis. That means we can look at the values of $0 \le \theta \le \pi$ and then reflect them over the polar axis.



This is also a limaçon.

Example: Describe the zeros and maximum r-values of the polar equation $r = 5\cos 2\theta$.

- Points where $|r|$ is maximum.

Since $-1 \le \cos 2\theta \le 1$, the maximum for $|r|$ will be:

$$\begin{aligned} r &= 5\cos 2\theta \\ r &= 5(1) \\ r &= 5 \\ |r| &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} r &= 5\cos 2\theta \\ r &= 5(-1) \\ r &= -5 \\ |r| &= 5 \end{aligned}$$

Therefore, the maximum value of $|r|$ is 5, and it occurs when

$$\begin{aligned} r &= 5\cos 2\theta \\ 5 &= 5\cos 2\theta \\ \cos 2\theta &= 1 \\ 2\theta &= 0, 2\pi, 4\pi, 6\pi\ldots \\ \theta &= 0, \pi, 2\pi, 3\pi,\ldots \end{aligned} \qquad \text{or} \qquad \begin{aligned} r &= 5\cos 2\theta \\ -5 &= 5\cos 2\theta \\ \cos 2\theta &= -1 \\ 2\theta &= \pi, 3\pi, 5\pi, 7\pi\ldots \\ \theta &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\ldots \end{aligned}$$

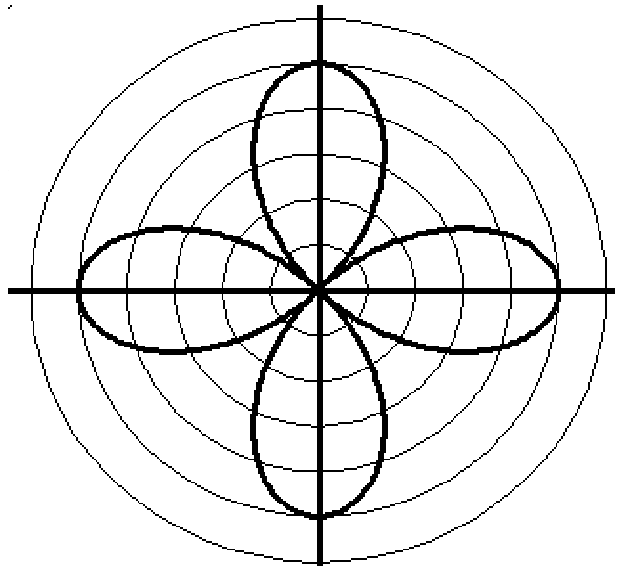
Since cosine is a periodic function, we only need to consider the values from 0 to 2π. Thus our maximum occurs when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

- The zeros of the equation.
  Set r = 0 and solve for θ.

$$r = 5\cos 2\theta$$
$$0 = 5\cos 2\theta$$
$$0 = \cos 2\theta$$
$$\cos 2\theta = 0$$
$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \ldots$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \ldots$$

Again, we only need to consider the values from 0 to 2π.
Thus our zeros occur when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

The graph of $r = 5\cos 2\theta$ is:



Sketching a Polar Graph

Example: Sketch the graph of $r = 1 - \sin\theta$.

Solution:

1. Symmetry: With respect to the line $\theta = \dfrac{\pi}{2}$.
2. Maximum: The max of |*r*| will happen when $\sin\theta = \pm 1$.

$$r = 1 - \sin\theta \qquad\qquad r = 1 - \sin\theta$$
$$r = 1 - (1) = 0 = 0 \quad \text{or} \quad r = 1 - (-1) = 2$$

The max value for r is 2 and happens when $\sin\theta = -1$. This happens when $\theta = \dfrac{3\pi}{2}$.

3. Zeros: Let r = 0 and solve.

$$r = 1 - \sin\theta$$

$$0 = 1 - \sin\theta$$

$$\sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

4. Sketch the graph in intervals. Make a table if needed.

Table for $r = 1 - \sin\theta$.

θ	r
0	1
$\frac{\pi}{6}$	0.5
$\frac{\pi}{4}$	0.3
$\frac{\pi}{3}$	0.1
$\frac{\pi}{2}$	0

θ	r
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	0.1
$\frac{3\pi}{4}$	0.3
$\frac{5\pi}{6}$	0.5
π	1

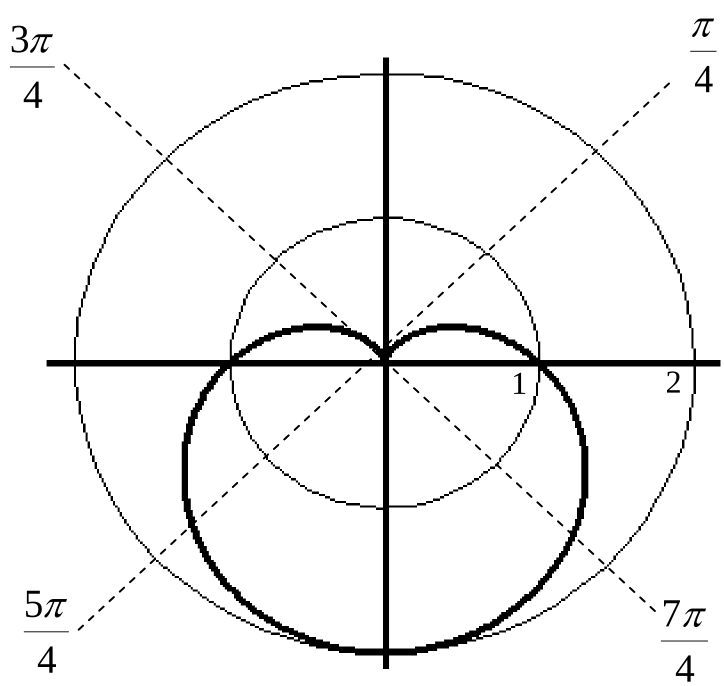
θ	r
π	1
$\frac{7\pi}{6}$	1.5
$\frac{5\pi}{4}$	1.7
$\frac{4\pi}{3}$	1.9
$\frac{3\pi}{2}$	2

θ	r
$\frac{3\pi}{2}$	2
$\frac{5\pi}{3}$	1.9
$\frac{7\pi}{4}$	1.7
$\frac{11\pi}{6}$	1.5
2π	1

Hint: Use the TABLE feature of your calculator for table values. At the TABLSET screen, choose "Ask" for the independent variable.

- As θ goes from 0 to $\frac{\pi}{2}$, r goes from 1 to 0.
- As θ goes from $\frac{\pi}{2}$ to π, r goes from 0 to 1.
- As θ goes from π to $\frac{3\pi}{2}$, r goes from 1 to 2.
- As θ goes from $\frac{3\pi}{2}$ to 2π, r goes from 2 to 1.

Sketch the graph.



Example: Sketch the graph of $r = \cos 2\theta$.

Solution:

1. Symmetry: With respect to the polar axis.
2. Maximum: Since the max of cosine is 1, $|r| = 1$. This happens when

$$\begin{array}{lcl} \cos 2\theta = 1 & & \cos 2\theta = -1 \\ 2\theta = 0, 2\pi, 4\pi, 6\pi, \ldots & \text{or} & 2\theta = \pi, 3\pi, 5\pi, 6\pi, \ldots \\ \theta = 0, \pi & & \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \end{array}$$

3. Zeros: Let r = 0 and solve.

$$0 = \cos 2\theta$$

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

4. Sketch the graph in intervals. Make a table if needed.

Table for $r = \cos 2\theta$.

θ	r
0	1
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	0
$\frac{\pi}{3}$	$-\frac{1}{2}$
$\frac{\pi}{2}$	-1

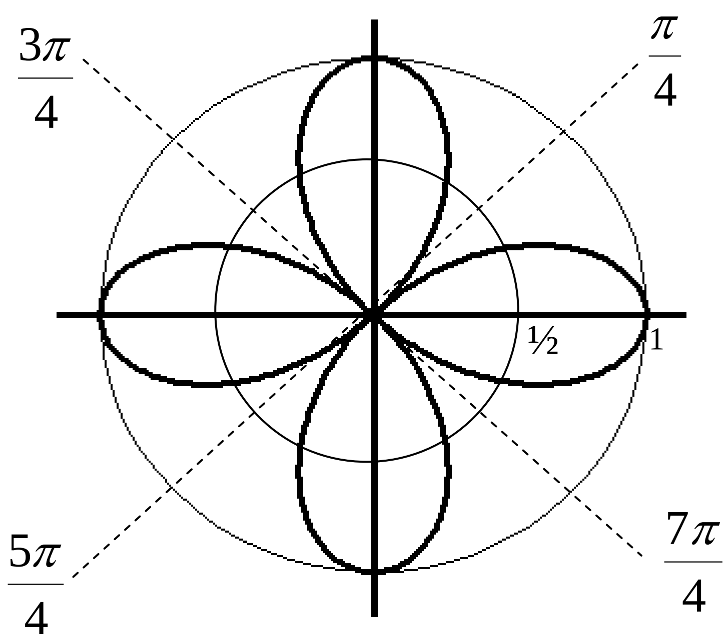
θ	r
$\frac{\pi}{2}$	-1
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	0
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	1

θ	r
π	1
$\frac{7\pi}{6}$	$\frac{1}{2}$
$\frac{5\pi}{4}$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{2}$	-1

θ	r
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{1}{2}$
$\frac{7\pi}{4}$	0
$\frac{11\pi}{6}$	$\frac{1}{2}$
2π	1

- As θ goes from 0 to $\frac{\pi}{4}$, r goes from 1 to 0.
- As θ goes from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r goes from 0 to -1.
- As θ goes from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, r goes from -1 to 0.
- As θ goes from $\frac{3\pi}{4}$ to π, r goes from 0 to 1.
- As θ goes from π to $\frac{5\pi}{4}$, r goes from 1 to 0.

- As θ goes from $\frac{5\pi}{4}$ to $\frac{3\pi}{2}$, r goes from 0 to -1.
- As θ goes from $\frac{3\pi}{2}$ to $\frac{7\pi}{4}$, r goes from -1 to 0.
- As θ goes from $\frac{7\pi}{4}$ to 2π, r goes from 0 to 1.



Graphing Polar Equations using a Graphing Calculator

1. Change the mode to Polar by pressing [MODE] and on the 4th line highlight [Pol].
2. Press [Y=] to see the equation editor for polar equations. Enter your equation and press [GRAPH].
3. If the graph does not fit in the window, press [ZOOM] [ZoomFit].

Special Polar Graphs

Look at the special polar graphs on page 760.