Polynomial and Rational Functions

Long Division of Polynomials

Divide $2x^3 - 5x^2 + x - 8$ by $x - 3$

$$\begin{array}{r} 2x^2 + x - 4 \\ x - 3 \overline{)\,2x^3 - 5x^2 + x - 8} \\ \underline{2x^3 - 6x^2} \\ x^2 + x \\ \underline{x^2 - 3x} \\ 4x - 8 \\ \underline{4x - 12} \\ 4 \end{array}$$

$$\begin{array}{r} 118 \\ 61\overline{)7213} \\ \underline{-61} \\ 111 \\ \underline{-61} \\ 503 \\ \underline{-488} \\ 15 \end{array}$$

$$\frac{7213}{61} = 118\,\frac{15}{61}$$

Note: To check this, we would multiply back
$(x - 3)(2x^2 + x + 4) + 4$

So, $\dfrac{2x^3 - 5x^2 + x - 8}{x - 3} = 2x^2 + x + 4 + \dfrac{4}{x - 3}$

Example: Divide $3x^3 - x^2 + 2x - 3$ by $x - 2$

$$\begin{array}{r l}
 & \quad 3x^2 + 5x + 12 \\
x - 2 & \big) \overline{3x^3 - x^2 + 2x - 3} \\
 & \underline{3x^3 - 6x^2} \\
 & \qquad\quad 5x^2 + 2x \\
 & \qquad\quad \underline{5x^2 - 10x} \\
 & \qquad\qquad\quad 12x - 3 \\
 & \qquad\qquad\quad \underline{12x - 24} \\
 & \qquad\qquad\qquad\quad 21
\end{array}$$

To check this, multiply back $(x - 2)(3x^2 + 5x + 12) + 21$

So, $\dfrac{3x^3 - x^2 + 2x - 3}{x - 2} = 3x^2 + 5x + 12 + \dfrac{21}{x - 2}$

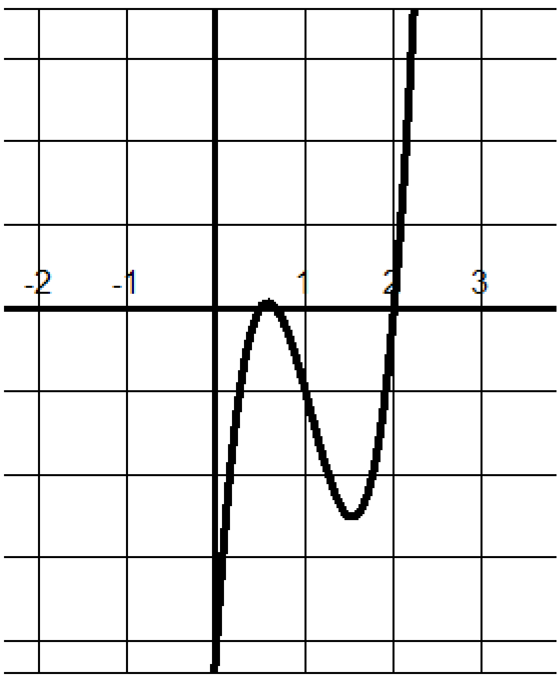
Example: Divide $3x^3 + 4x - 2$ by $x^2 + 2x + 1$

$$\begin{array}{r l}
 & \qquad\qquad\quad 3x - 6 \\
x^2 + 2x + 1 & \big) \overline{3x^3 - 0x^2 + 4x - 2} \\
 & \underline{3x^3 + 6x^2 + 3x} \\
 & \qquad\quad -6x^2 + 1x - 2 \\
 & \qquad\quad \underline{-6x^2 - 12x - 6} \\
 & \qquad\qquad\quad 13x + 4
\end{array}$$

So, $\dfrac{3x^3 + 4x - 2}{x^2 + 2x + 1} = 3x - 6 + \dfrac{13x + 4}{x^2 + 2x + 1}$

Using Long Division to find Zeros

Look at $f(x) = 6x^3 – 19x^2 + 16x – 4$



Suppose you knew one of the zeros is $x = 2$.

Then you know that when you are solving algebraically to find zeros you must have

$(x-2)($ *something* $) = 0$

This means that $f(x) = (x-2)($ *something* $)$

or

$6x^3 – 19x^2 + 16x – 4 = (x-2)($ *something* $)$

$$\begin{array}{r|l} & 6x^2 - 7x + 2 \\ \hline x - 2 & 6x^3 - 19x^2 + 16x - 4 \\ & \underline{6x^3 - 12x^2} \\ & \quad -7x^2 + 16x \\ & \quad \underline{-7x^2 + 14x} \\ & \qquad 2x - 4 \\ & \qquad \underline{2x - 4} \\ & \qquad\quad 0 \end{array}$$

We use long division to find the “something.”

This means that $6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$

$$f(x) = (x - 2)(6x^2 - 7x + 2)$$

Let f(x) = 0 and continue to factor to find all zeros.

$$0 = (x - 2)(6x^2 - 7x + 2)$$
$$0 = (x - 2)(2x - 1)(3x - 2)$$
$$x - 2 = 0 \text{ or } 2x - 1 = 0 \text{ or } 3x - 2 = 0$$
$$x = 2 \text{ or } x = \frac{1}{2} \text{ or } x = \frac{2}{3}$$

So the zeros are 2, $\frac{1}{2}$, and $\frac{2}{3}$

**We were able to divide out the (x – 2) to get a polynomial that we could factor, and thus find the other zeros.

The Division Algorithm

For all polynomials *f*(*x*) and *d*(*x*) such that the degree of *d* is less than or equal to the degree of *f* and *d*(*x*) ≠ 0, there exist unique polynomials *q*(*x*) and *r*(*x*) such that

$$f(x) = d(x)q(x) + r(x)$$

dividend — divisor — quotient — remainder

where *r*(*x*)= 0 or the degree of *r* is less than the degree of *d*. If *r(x)* = 0, then *d(x)* divides evenly into *f(x).*

Notation: $\dfrac{f(x)}{d(x)} = q(x) + \dfrac{r(x)}{d(x)}$

Example: Divide $6x^3 - 3x^2 + 2x - 1$ by $x - 1$

$$
\begin{array}{r}
6x^2 + 3x + 5 \\
x - 1 \,\overline{)\, 6x^3 - 3x^2 + 2x - 1} \\
\underline{6x^3 - 6x^2} \qquad\qquad \\
3x^2 + 2x \qquad \\
\underline{3x^2 - 3x} \qquad \\
5x - 1 \\
\underline{5x - 5} \\
4
\end{array}
$$

We have $6x^3 - 3x^2 + 2x - 1 = (x - 1)(6x^2 + 3x + 5) + 4$

$f(x) = d(x) \bullet q(x) + r(x)$

(dividend) = (divisor)(quotient) + remainder

Notes on Long Division:

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variables.

Example: Find $(2x^4 + 4x^3 - 5x^2 + 3x - 2)$) $(x^2 + 2x - 3)$

$$
\begin{array}{r l}
 & 2x^2 + 0x + 1 \\
x^2 + 2x - 3 \,\big) & 2x^4 + 4x^3 - 5x^2 + 3x - 2 \\
 & \underline{2x^4 + 4x^3 - 6x^2} \\
 & \quad 0x^3 + 1x^2 + 3x \\
 & \quad \underline{0x^3 + 0x^2 + 0x} \\
 & \qquad x^2 + 3x - 2 \\
 & \qquad \underline{x^2 + 2x - 3} \\
 & \qquad\qquad x + 1
\end{array}
$$

We saw in a previous example that

$f(x) = 6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2)$

Dividing out $(x - 2)$ gave us a polynomial that we could factor to find the other zeros.

<u>Question:</u> What could you have done with $f(x) = 6x^3 - 19x^2 + 16x - 4$ if you were told to find the zeros but not given any more information?

Started doing long division with various divisors until we found one that worked.

Example: To find the zeros of $f(x) = 2x^3 – 1x^2 – x + 6$ we could start dividing by (x+1), (x+2), (x-3), (x-1), etc. until we found one that divided evenly.

Eventually, we would find that $f(x) = (x-1)(2x^2 + x – 6)$.

We would continue to factor to get $f(x) = (x-1)(2x-3)(x+2)$.

*To divide these more quickly we will use <u>synthetic division</u>.

<u>Synthetic Division</u>

Find $(3x^3 – x^2 + 2x – 3)$) $(x – 2)$

2	3	-1	2	-3
		6	10	24
	3	5	12	21

Bring the 3 down, then multiply by 2 to get 6, add to -1 to get 5, multiply by 2 to get 10, add to 2 to get 12, multiply by 2 to get 24, and add to -3 to get 21. The number 21 is the remainder.

Put the variable terms back in to get the solution:

$$3x^3 – x^2 + 2x – 3 = (x – 2)(3x^2 + 5x + 12) + 21$$

**Note: We can only use synthetic division if the divisor is (x-k).

Example: Divide $x^4 + 3x^3 + 2x - 1$ by $x+2$.

-2	1	3	0	2	-1
		-2	-2	4	-12
	1	1	-2	6	-13

$1x^3 + 1x^2 - 2x + 6$ Remainder -13

Example: Divide $2x^4 + 5x^2 - 3$ by $x - 5$.

5	2	0	5	0	-3
		10	50	275	1375
	2	10	55	275	1372

We get $2x^3 + 10x^2 + 55x + 275 + \frac{1372}{x-5}$

Look again at our previous example of

$$f(x) = 2x^3 - 1x^2 - 7x + 6$$

Use synthetic division to find a zero.

2	2	-1	-7	6
		4	6	-2
	2	3	-1	4

The remainder ≠ 0 so 2 <u>is not</u> a zero.

1	2	-1	-7	6
		2	1	-6
	2	1	-6	0

The remainder = 0 so 1 <u>is</u> a zero.

So we know we have :

$$f(x) = 2x^3 - 1x^2 - 7x + 6 = (x - 1)(2x^2 + 1x - 6)$$

Now we can factor the quadratic to find the remaining zeros.

$$f(x) = 2x^3 - 1x^2 - 7x + 6 = (x - 1)(x + 2)(2x - 3)$$

Find the zeros of $f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$

2	1	5	5	-5	-6
		2	14	38	66
	1	7	19	33	60

The remainder ≠ 0 so 2 <u>is not</u> a zero.

1	1	5	5	-5	-6
		1	6	11	6
	1	6	11	6	0

The remainder = 0 so 1 <u>is</u> a zero.

We now have $x^4 + 5x^3 + 5x^2 - 5x - 6 = (x-1)(x^3+6x^2+11x+6)$

Repeat the process with $x^3+6x^2+11x+6$

-2	1	6	11	6
		-2	-8	-6
	1	4	3	0

So $x^3+6x^2+11x+6 = (x+2)(x^2 + 4x + 3)$. Substitute back:

$$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6 = (x-1)(x^3+6x^2+11x+6)$$
$$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6 = (x-1)(x+2)(x^2 + 4x + 3)$$

Finish by factoring the quadratic:

$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6 = (x-1)(x+2)(x+1)(x+3)$

So our zeros are 1, -2, -1, and -3

(**notice**: these are all factors of our constant term in the original polynomial.)

** The remainder obtained in the synthetic division process has an important interpretation.

<u>The Remainder Theorem</u>:

If a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

**This means that we can use synthetic division as a shortcut to evaluating a polynomial. Instead of plugging k into the function, we can divide by (x-k) synthetically and our reminder will be f(k).

Example: Use the Remainder Theorem to find f(1) for $f(x) = x^3 - 2x^2 - 4x + 1$

1	1	-2	-4	1
		1	-1	-5
	1	-1	-5	-4

So, f(1) = -4

(1, -4) is a point of the graph of the function.

Example: Use the Remainder Theorem to find f(2) for $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$

2	2	7	-4	-27	-18
		4	22	36	18
	2	11	18	9	0

So, f(2) = 0

This means that 2 is a zero and (2, 0) is an x-intercept.