Rational Functions

<u>Definition</u> – A rational function can be written in the form $f(x) = \frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

*To find the <u>domain</u> of a rational function we must look at what values make the denominator = 0. These numbers must be *excluded* from the domain.

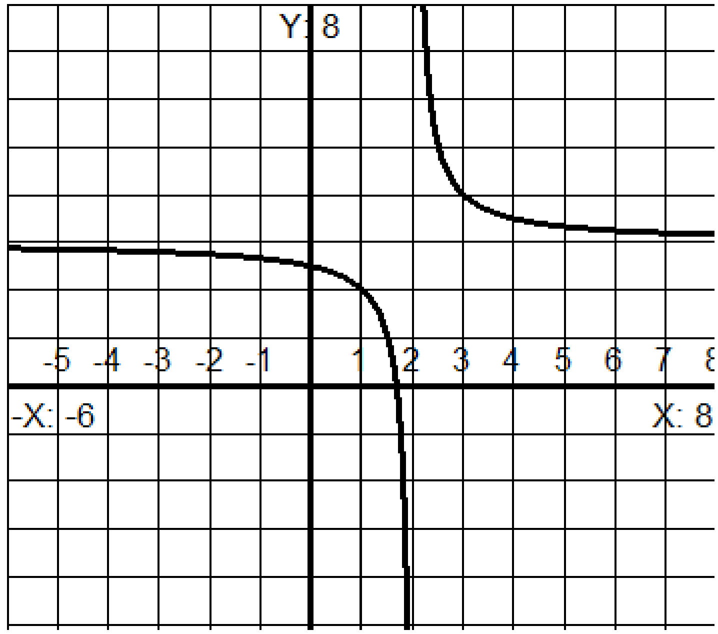
Example: Find the domain of $f(x) = \frac{1}{x^2 - 9}$.

Set $x^2 - 9 = 0$ and solve.
$(x - 3)(x + 3) = 0$
$x = 3$ or $x = -3$

We say the domain is all real numbers except 3 and -3.

We can also list it as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

Look at $f(x)=\dfrac{3x-5}{x-2}$.



Questions:

What happens to y as x approaches 2 from the left?

y approaches - ∞

What happens to y as x approaches 2 from the right?

y approaches + ∞

What happens to y as x approaches ∞?

y approaches 3

What happens to y as x approaches - ∞?

y approaches 3

Notation: $f(x) \to -\infty$ as $x \to 2^-$
$f(x) \to \infty$ as $x \to 2^+$
$f(x) \to 3$ as $x \to -\infty$
$f(x) \to 3$ as $x \to \infty$

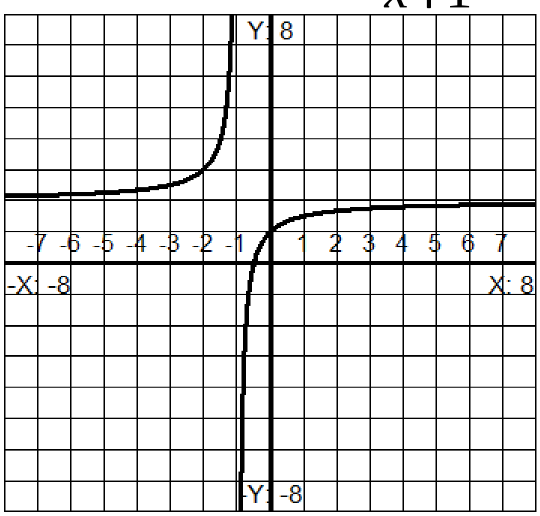
*The line x = 2 is called the vertical asymptote.
*The line y = 3 is called the horizontal asymptote.

Definition of asymptotes:

1. The line x = a is a vertical asymptote of the graph of *f* if $f(x) \to \pm\infty$ as $x \to a$, either from the right or from the left.

2. The line y = b is a horizontal asymptote of the graph of *f* if $f(x) \to b$ as $x \to \pm\infty$.

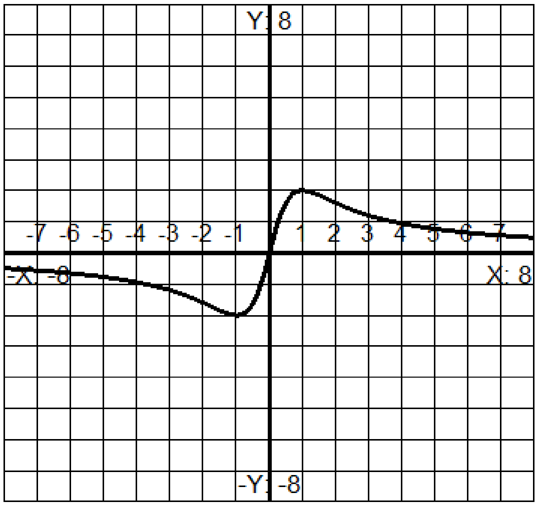
Look at: $f(x) = \dfrac{2x+1}{x+1}$



The horizontal asymptote is
y = 2

The vertical asymptote is
x = -1

Look at $f(x)=\dfrac{4x}{x^2+1}$

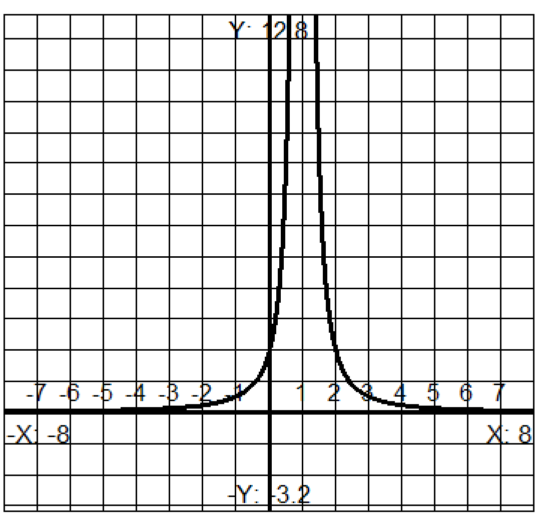


The horizontal asymptote is y = 0

There is no vertical asymptote.

*Note that the graph crosses the asymptote at the origin. Sometimes the graph of a rational function will cross asymptotes at or around the origin.

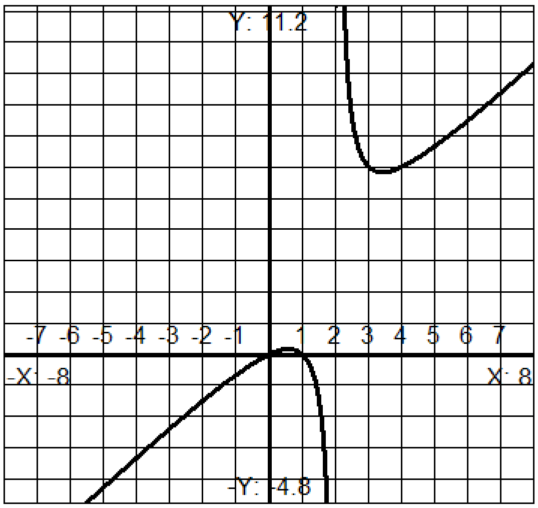
Look at $f(x)=\dfrac{2}{(x-1)^2}$



The horizontal asymptote is y = 0

The vertical asymptote is x = 1

Look at $f(x) = \dfrac{x^2 - x}{x-2}$



There is no horizontal asymptote.

The vertical asymptote is x = 2

Rules for Asymptotes of Rational Functions:

Let *f* be a rational function given by

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{n-1} + \ldots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of *f* has *vertical* asymptotes at the zeros of the polynomial *D*(*x*). (ie. where *D(x)* = 0)

2. The graph of *f* has one or no ***horizontal*** asymptote, depending on the degree of *N* and *D*.

   a. If $n < m$, then $y = 0$ is the horizontal asymptote of the graph of *f*.

   b. If $n = m$, then $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the graph of *f*.

   c. If $n > m$, then the graph of *f* has no horizontal asymptote.

Examples: Find the horizontal and vertical asymptotes.

a) $f(x) = \dfrac{2x+5}{4x-6}$

horizontal asymptote: $y = \frac{1}{2}$

vertical asymptote: $x = \frac{3}{2}$

b) $$f(x) = \frac{1}{x-2}$$

horizontal asymptote: y = 0
vertical asymptote: x = 2

c) $$f(x) = \frac{x^2}{x+1}$$

horizontal asymptote: none
vertical asymptote: x = -1

d) $$f(x) = \frac{2x-1}{x^2-x-6}$$

$$f(x) = \frac{2x-1}{(x-3)(x+2)}$$

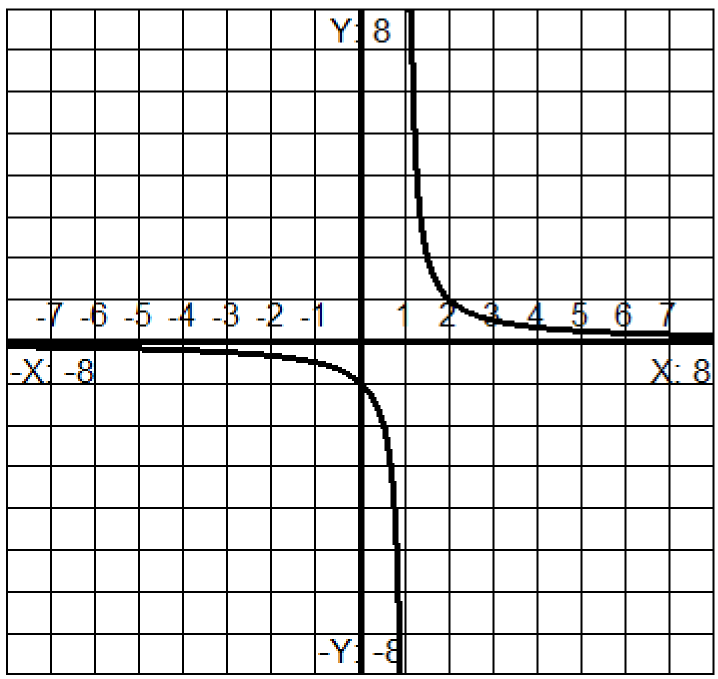
horizontal asymptote: y=0
vertical asymptote: x = 3 and x = -2

Look at: $f(x)=\dfrac{x+1}{x^2-1}$

What is the domain?

all real numbers except 1 and -1

Look at the graph on your calculator:



Is there an asymptote at x = -1?

No

On your calculator, look at the table values for 1 and -1.

[TblSet] TblStart = -2 and the [Table]

What values are given for 1 and -1?

Both show "error."

Look at $f(x) = \dfrac{x+1}{x^2-1}$ again. Factor it to

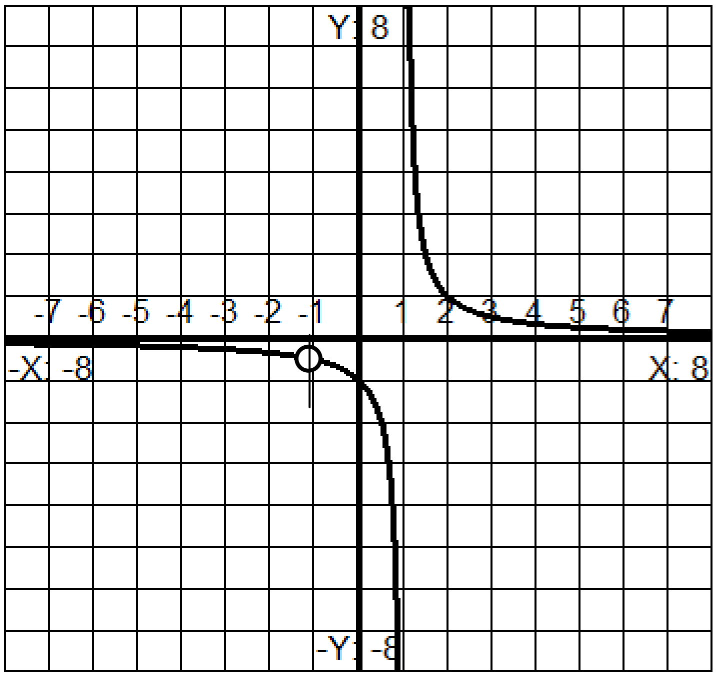
$$f(x) = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}$$

Graph $f(x) = \dfrac{1}{x-1}$ and you will get the same graph.

**Since $f(x) = \dfrac{x+1}{x^2-1}$ is our original equation,

we cannot ignore our original domain, even if the equation simplifies to something else. Because x ≠ -1, we will put a "hole" at x = -1.

The graph of $f(x)=\frac{x+1}{x^2-1}$ should look like:



There is a "hole" at x = -1.

Guidelines for Analyzing Graphs of Rational Functions:

Let $f(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials with no common factors.

1) Find and plot the y-intercept (if any) by evaluating $f(0)$.

2) Find the zeros of the numerator (if any) by solving the equation $N(x) = 0$. Then plot the corresponding x-intercepts.

3) Find the zeros of the denominator (if any) by solving the equation $D(x) = 0$. Then sketch the corresponding vertical asymptotes.

4) Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.

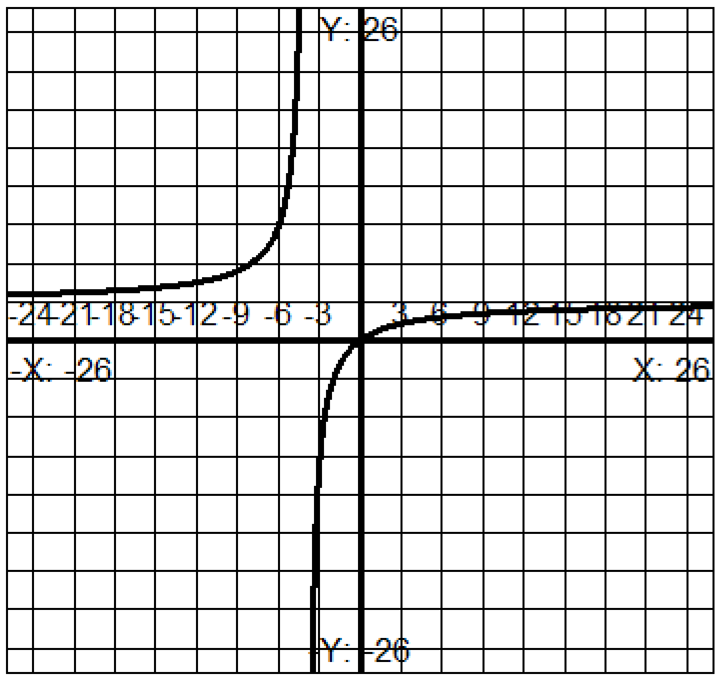
5) Test for symmetry.

6) Plot at least one point between and one point beyond each x-intercept and vertical asymptote.

7) Use smooth curves to complete the graph between and beyond the vertical asymptotes.

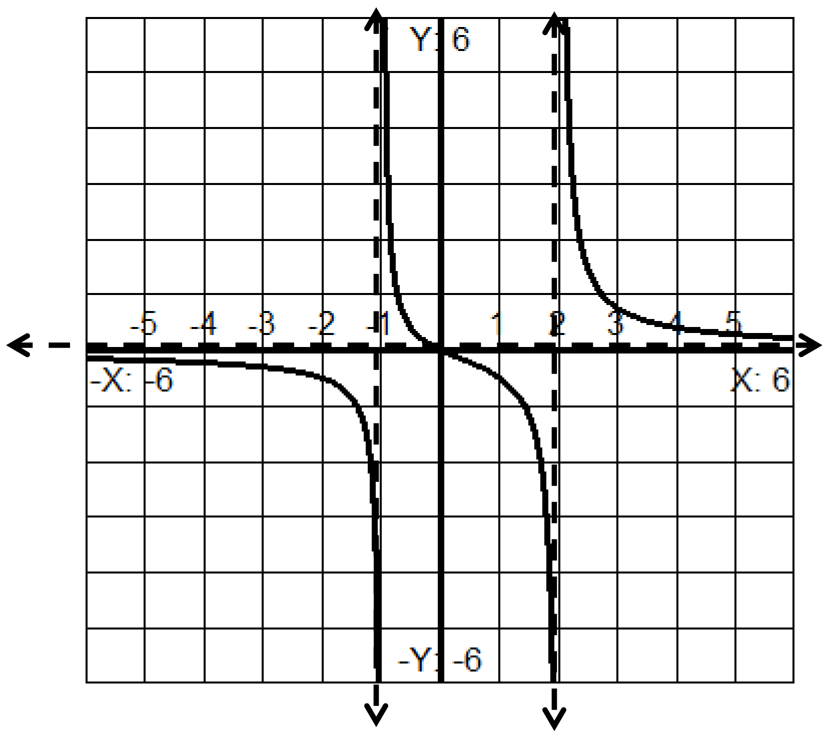
Note: Because the function can only change signs at its zeros and vertical asymptotes, we use these values to determine test intervals.

Example: Sketch $f(x)=\frac{3x}{x+4}$.



y-intercept: (0,0)
x-intercept: (0,0)
vertical asymptote: x = -4
horizontal asymptote: y = 3
Points in test intervals: (-5, 15)
(-2, -3)
(2, 1)

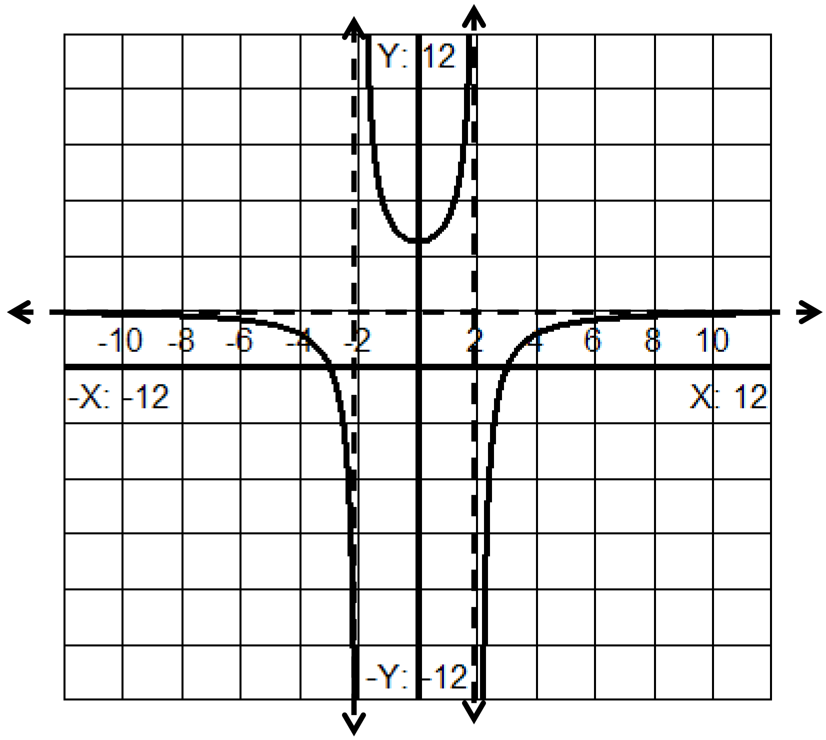
Example: Sketch $f(x)=\dfrac{x}{x^2-x-2}$.



y-intercept: (0,0)
x-intercept: (0,0)
vertical asymptote: x = 2, x = -1
horizontal asymptote: y = 0
Points in test intervals: (-3, -0.3)
(-0.5, 0.4)
(1, -0.5)
(3, 0.75)

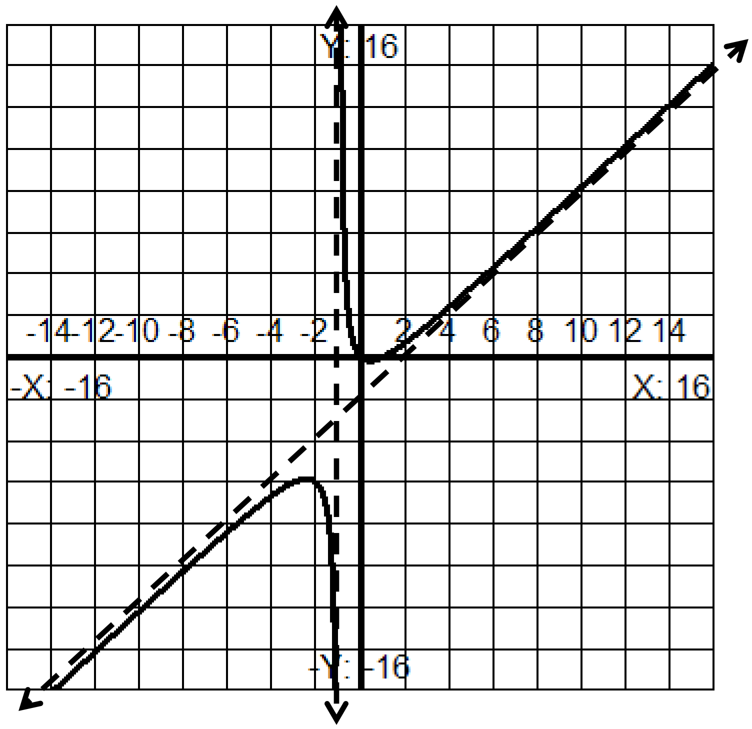
Example: Sketch $f(x)=\frac{2(x^2-9)}{x^2-4}$

$$f(x)=\frac{2(x^2-9)}{x^2-4}=\frac{2(x-3)(x+3)}{(x-2)(x+2)}$$



y-intercept: (0,4.5)
x-intercept: (-3,0) and (3,0)
vertical asymptote: x = 2, x = -2
horizontal asymptote: y = 2
symmetry: y-axis because it is an even function
Points in test intervals: (-6, 1.69)
(-2.5, -2.44)
(0.5, 4.67)
(2.5, -2.44)
(6, 1.69)

Look at: $f(x)=\dfrac{x^2-x}{x+1}$



*This function has a slant asymptote. This happens only if the degree of the numerator is *exactly* one more than the degree of the denominator.

To Find Slant Asymptotes:

Use long division to divide the denominator into the numerator. The equation of the asymptote is the quotient, excluding the remainder.

Look again at $f(x) = \frac{x^2 - x}{x+1}$.

Do long division:

$$\begin{array}{r} x - 2 \\ x+1 \overline{)\, x^2 - x + 0} \\ \underline{x^2 + x} \\ -2x + 0 \\ \underline{-2x - 2} \\ 2 \end{array}$$

So we have:

$$f(x) = \frac{x^2 - x}{x+1} = x - 2 + \frac{2}{x+1}$$

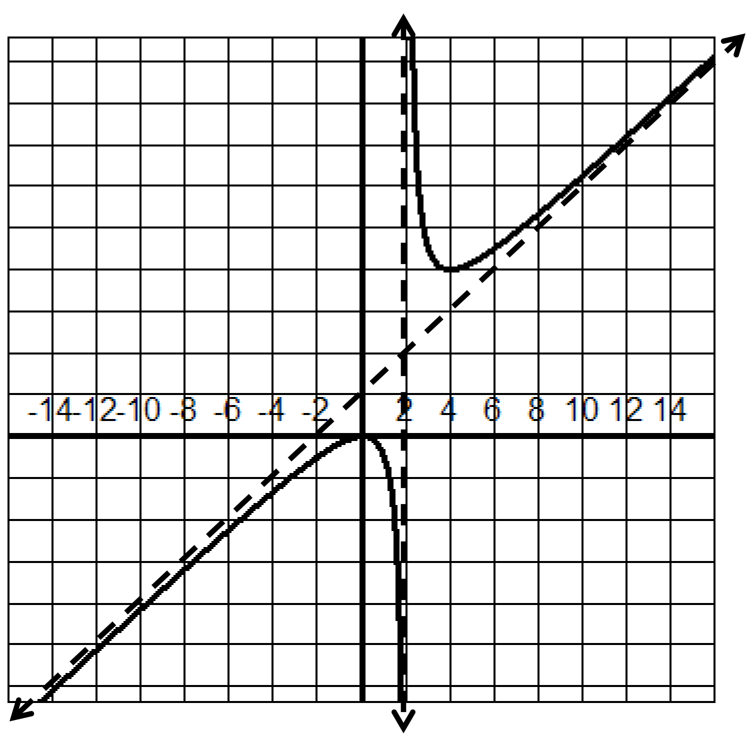
Then the slant asymptote is

$$f(x) = x - 2$$

(or $y = x - 2$)

Example: Sketch $f(x) = \dfrac{x^2}{x-2}$

y-intercept: (0,0)
x-intercept: (0,0)
vertical asymptote: x=2
slant asymptote: y=x+2
Points: (-1/2, -0.1)
(1, -1)
(3, 9)



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where $N(x)$ and $D(x)$ have no common factors.

3. The graph of *f* has ***vertical*** asymptotes at the zeros of the polynomial $D(x)$. (ie. where $D(x) = 0$)

4. The graph of *f* has one or no ***horizontal*** asymptote, depending on the degree of N and D.

   a. If $n < m$, then $y = 0$ is the horizontal asymptote of the graph of *f*.

   b. If $n = m$, then $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the graph of *f*.

   c. If $n > m$, then the graph of *f* has no horizontal asymptote.

5. If n=m+1, then the graph of *f* has a ***slant*** asymptote at y=*q*(x), where *q*(x) is the quotient obtained from the division algorithm, excluding any remainder.