Properties of Logarithms

Change of Base

Most calculators only have LOG and LN buttons. To evaluate logs of other bases we use the change-of-base formula.

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then

$$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$$

Examples: Evaluate the following.

(a) $\log_5 18$

$$\frac{\log 18}{\log 5} \approx 1.7959 \quad \text{or} \quad \frac{\ln 18}{\ln 5} \approx 1.7959$$

(b) $\log_2 42$

$$\frac{\log 42}{\log 2} \approx 5.3923 \quad \text{or} \quad \frac{\ln 42}{\ln 2} \approx 5.3923$$

(c) $\log_3 14$

$$\frac{\log 14}{\log 3} \approx 2.4022 \quad \text{or} \quad \frac{\ln 14}{\ln 3} \approx 2.4022$$

Properties of Logarithms

Since logarithms are exponents, they will follow rules similar to exponent properties.

Properties of Logarithms

Let a be a positive real number such that $a \neq 1$, let n be a real number, and let u and v be positive real numbers. Then

1. $\log_a(uv) = \log_a u + \log_a v$
2. $\log_a \left(\frac{u}{v}\right) = \log_a u - \log_a v$
3. $\log_a u^n = n\log_a u$

1. $\ln(uv) = \ln u + \ln v$
2. $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$
3. $\ln u^n = n\ln u$

Examples: Write each logarithm in terms of ln 2 and ln 3.

(a) $\ln 6$

$= \ln(2 \cdot 3)$

$= \ln 2 + \ln 3$

(b) $\ln \frac{2}{81}$

$= \ln 2 - \ln 81$

$= \ln 2 - \ln 3^4$

$= \ln 2 - 4\ln 3$

Example:

Verify that $-\log_{10}\frac{1}{100}=\log_{10}100$

Solution:

$$
\begin{aligned}
-\log_{10}\frac{1}{100} &= -\log_{10}\frac{1}{100} \\
&= -\left(\log_{10}100^{-1}\right) \\
&= -[(-1)\log_{10}100] \\
&= \log_{10}100
\end{aligned}
$$

⟵ Always start a verification with a statement like this.

Rewriting Logarithmic Expressions

To expand a logarithmic expression means to use the properties of logarithms to rewrite the expression as a sum, difference, and/or constant multiple of logarithms.

Examples: Expand each logarithmic expression.

(a) $\log(2x^3y^4)$

$$=\log 2+\log x^3+\log y^4$$

$$=\log 2+3\log x+4\log y$$

(b) $\log \frac{4x^2}{y^3}$

$$= \log 4x^2 - \log y^3$$
$$= \log 4 + \log x^2 - 3\log y$$
$$= \log 4 + 2\log x - 3\log y$$

(c) $\ln \frac{\sqrt{x+5}}{y^2}$

$$= \ln \sqrt{x+5} - \ln y^2$$
$$= \ln(x+5)^{\frac{1}{2}} - 2\ln y$$
$$= \frac{1}{2}\ln(x+5) - 2\ln y$$

(d) $\ln \frac{xy^4}{2}$

$$= \ln xy^4 - \ln 2$$
$$= \ln x + \ln y^4 - \ln 2$$
$$= \ln x + 4\ln y - \ln 2$$

To condense a logarithmic expression means to use the properties of logarithms to rewrite the expression as the logarithm of a single quantity.

Examples: Condense each logarithmic expression.

(a) $2\log x - 3\log y + \frac{1}{2}\log z$

$$= \log x^2 - \log y^3 + \log \sqrt{z}$$
$$= \log x^2 + \log \sqrt{z} - \log y^3$$
$$= \log x^2\sqrt{z} - \log y^3$$
$$= \log \frac{x^2\sqrt{z}}{y^3}$$

(b) $\frac{1}{3}\left(2\ln x - 4\ln y - \ln(z+2)\right)$

$$= \frac{1}{3}\left(\ln x^2 - \ln y^4 - \ln(z+2)\right)$$
$$= \frac{1}{3}\left(\ln \frac{x^2}{y^4(z+2)}\right)$$
$$= \ln \sqrt[3]{\frac{x^2}{y^4(z+2)}}$$

(c) $2\ln(x+2)-\frac{1}{2}\ln x$

$$= \ln(x+2)^2 - \ln x^{\frac{1}{2}}$$
$$= \ln(x+2)^2 - \ln\sqrt{x}$$
$$= \ln\frac{(x+2)^2}{\sqrt{x}}$$

(d) $3\log x + 4\log(x-1)$

$$= \log x^3 + \log(x-1)^4$$
$$= \log x^3(x-1)^4$$

Application:

One way of finding a model for a set of nonlinear data is to take the natural log of each of the *x*-values and *y*-values of the data set. If the points are graphed and fall on a straight line, then the *x*-values and the *y*-values are related by the equation: $\ln y = m \ln x$, where *m* is the slope of the straight line.

Application

One method of determining how the x- and y-values for a set of nonlinear data are related begins by taking the natural log of each of the x- and y-values. If the points are graphed and fall on a straight line, then you can determine that the x- and y-values are related by the equation

$$\ln y = m \ln x$$

where m is the slope of the straight line.

Example 7 Finding A Mathematical Model

The table shows the mean distance x and the period (the time it takes a planet to orbit the sun) y for each of the six planets that are closest to the sun. In the table, the mean distance is given in terms of astronomical units (where Earth's mean distance is defined as 1.0), and the period is given in terms of years. Find an equation that expresses y as a function of x.

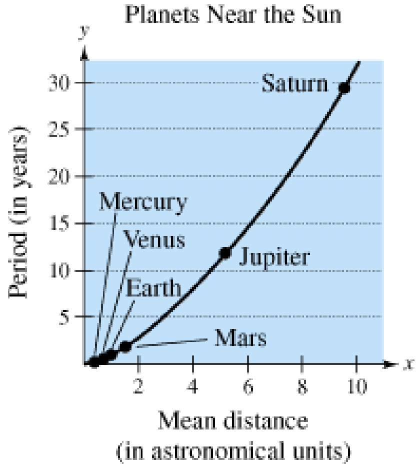

FIGURE 24

Planet	Mean distance, x	Period, y
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.862
Saturn	9.555	29.458

Solution

The points in the table are plotted in Figure 24. From this figure it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values in the table. This produces the following results.

Planet	$\ln x$	$\ln y$
Mercury	-0.949	-1.423
Venus	-0.324	-0.486
Earth	0.000	0.000
Mars	0.421	0.632
Jupiter	1.649	2.473
Saturn	2.257	3.383

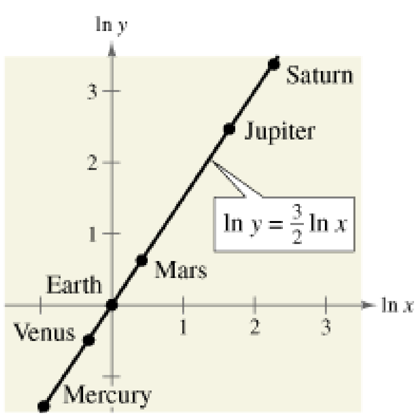

FIGURE 25

Now, by plotting the points in the second table, you can see that all six of the points appear to lie in a line (see Figure 25). You can use a graphical approach or the algebraic approach discussed earlier to find that the slope of this line is $\frac{3}{2}$. You can therefore conclude that $\ln y = \frac{3}{2} \ln x$.

Example: On the Richter scale, the magnitude *R* of an earthquake of intensity *I* is given by

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where I_0 is the minimum intensity used for comparison. Write this as a single common logarithmic expression.

$$R = \frac{\frac{\log I}{\log e} - \frac{\log I_0}{\log e}}{\frac{\log 10}{\log e}}$$

$$R = \frac{\frac{\log I - \log I_0}{\log e}}{\frac{\log 10}{\log e}}$$

$$R = \frac{\log I - \log I_0}{\log e} \div \frac{\log 10}{\log e}$$

$$R = \frac{\log I - \log I_0}{\log e} \cdot \frac{\log e}{\log 10}$$

$$R = \frac{\log I - \log I_0}{\log 10}$$

$$R = \frac{\log I - \log I_0}{1}$$

$$R = \log I - \log I_0$$

$$R = \log \frac{I}{I_0}$$