Exponential and Logarithmic Equations

*There are 2 basic strategies for solving exponential and logarithmic equations. One is based on the ***One-to-One Properties*** and the second is based on the ***Inverse Properties.***

One-to-One Properties

1. $a^x = a^y$ if and only if $x = y$.
2. $\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

1. $\log_a a^x = x$
2. $a^{\log_a x} = x$

Strategies for solving Exponential and Logarithmic Equations:

- Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions to solve the equation.
- Rewrite an exponential equation in logarithmic form and apply the Inverse Property of logarithmic functions to solve the equation.
- Rewrite a logarithmic equation in exponential form and apply the Inverse Property of exponential functions to solve the equation.

Examples using One-to-One Properties: Solve.

(a) $2^x = 64$

$2^x = 2^6$

Set the exponents equal.

$x = 6$

(b) $\left(\frac{1}{3}\right)^x = 9$

$\left(\frac{1^x}{3^x}\right) = 9, \quad \frac{1}{3^x} = 9$

$3^{-x} = 3^2$

Set the exponents equal.

$-x = 2$

$x = -2$

(c) $\ln x - \ln 3 = 0$

$\ln x = \ln 3$

Set arguments equal.

$x = 3$

(d) $5^x = 0.04$

$5^x = \frac{1}{25}$

$5^x = 5^{-2}$

$x = -2$

<u>To Solve Using One-to-One Properties:</u>

1. Isolate the variable term.
2. If it is *exponential*, and the bases are the same, set the exponents equal and solve.
3. If it is *logarithmic* and the bases are the same, set the arguments equal and solve.

Solving Exponential Equations Using the Inverse Properties

1. Isolate the exponential expression.
2. Take the logarithm of each side.
3. Apply the Inverse Properties.

*Note: When taking the log of each side, use the same base log as the base on the exponential expression.

Solving Logarithmic Equations Using the Inverse Properties

1. Isolate the logarithmic expression.
2. Write each side in exponential form.
3. Apply the Inverse Properties.

*Note: Use the same base as the base on the logarithmic expression.

Examples: Solve the following.

(a) $e^x = 11$

Take the log of each side

$\ln e^x = \ln 11$

$x = \ln 11 \approx 2.398$

(b) $\ln x = -3$

Use e as base on both sides.

$e^{\ln x} = e^{-3}$

$x = e^{-3} \approx 0.0498$

(c) $10^x = 90$

$$\log 10^x = \log 90$$
$$x = \log 90 \approx 1.954$$

(d) $\log x = -1$

$$10^{\log x} = 10^{-1}$$
$$x = 10^{-1} = \frac{1}{10}$$

(e) $4e^{2x} = 16$

$$e^{2x} = 4$$
$$\ln e^{2x} = \ln 4$$
$$2x = \ln 4$$
$$x = \frac{\ln 4}{2}$$
$$x \approx 0.693$$

(f) $5e^{x+2} - 8 = 14$

$$5e^{x+2} = 22$$
$$e^{x+2} = \frac{22}{5}$$
$$\ln e^{x+2} = \ln \frac{22}{5}$$
$$x + 2 = 1.48$$
$$x = 1.48 - 2 \approx -0.518$$

(g) $4^x = 30$

<u>There are 2 ways to do this problem:</u>

1. Take the $\log_4$ of each side:

$$\log_4 4^x = \log_4 30$$
$$x = \log_4 30$$
$$\text{Use change-of-base formula.}$$
$$x = \frac{\log 30}{\log 4} \approx 2.453$$

2. Take the log *or* ln of both sides and apply the Property of Exponents.

$$\log 4^x = \log 30$$
$$x \log 4 = \log 30$$
$$x = \frac{\log 30}{\log 4} \approx 2.453$$

<u>or</u>

$$\ln 4^x = \ln 30$$
$$x \ln 4 = \ln 30$$
$$x = \frac{\ln 30}{\ln 4} \approx 2.453$$

****You can use either method.**

(h) $2\left(3^x - 1\right) = 10$

$$3^x - 1 = 5$$
$$3^x = 6$$
$$\log_3 3^x = \log_3 6$$
$$x = \log_3 6$$
$$x = \frac{\log 6}{\log 3} \approx 1.631$$

<u>or</u>

$$3^x - 1 = 5$$
$$3^x = 6$$
$$\ln 3^x = \ln 6$$

Use property of exponents.

$$x \ln 3 = \ln 6$$
$$x = \frac{\ln 6}{\ln 3} \approx 1.631$$

(i) $e^{2x} - e^x - 20 = 0$

Think of this as: $(e^x)^2 - (e^x) - 20 = 0$

This is just a quadratic equation that can be factored.

$$(e^x)^2 - (e^x) - 20 = 0$$
$$(e^x - 5)(e^x + 4) = 0$$

$e^x = 5$ or $e^x = -4$

$\ln e^x = \ln 5$ or $\ln e^x = \ln(-4)$

$$x = \ln 5 \approx 1.609$$

*<u>Note</u>: ln(-4) is impossible so the only solution is ln 5.

(j) $e^{x-2} - 7 = 59$

$$e^{x-2} = 66$$
$$\ln e^{x-2} = \ln 66$$
$$x - 2 = \ln 66$$
$$x = (\ln 66) + 2$$
$$x \approx 6.190$$

(k) $2\log x = 5$

$$\log x = \frac{5}{2}$$
$$10^{\log x} = 10^{\frac{5}{2}}$$
$$x = 10^{\frac{5}{2}} = \sqrt{10^5} \approx 316.228$$

(l) $4\ln 5x = 28$

$$\ln 5x = 7$$
$$e^{\ln 5x} = e^7$$
$$5x = 7$$
$$x = \frac{7}{5}$$

(m) $\ln\sqrt{x+2} = \ln x$

The bases are the same, so set the arguments equal.

$$\sqrt{x+2} = x$$
$$\left(\sqrt{x+2}\right)^2 = x^2$$
$$x+2 = x^2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \quad \textbf{or} \quad x = -1$$

Check both in the original equation:

<u>Check x=2</u>

$$\ln\sqrt{(2)+2} = \ln(2)$$
$$\ln\sqrt{4} = \ln 2$$
$$\ln 2 = \ln 2$$

<u>Check x= -1</u>

$$\ln\sqrt{(-1)+2} = \ln(-1)$$

You can stop here, because you cannot do ln(-1). So x = -1 is an <u>extraneous solution</u>.

(n) $\log x - \log(x-3) = 1$

$$\log \frac{x}{x-3} = 1$$

← Use the Quotient Property for Logarithms.

$$10^{\log \frac{x}{x-3}} = 10^1$$

$$\frac{x}{x-3} = 10$$

$$x = 10(x-3)$$

$$x = 10x - 30$$

$$-9x = -30$$

$$x = \frac{30}{9} = \frac{10}{3}$$

Solving an Equation on a Graphing Calculator

When solving any equation, you can do it graphically by graphing the left and right sides separately on your graphing calculator, and then using the intersect feature to determine the point of intersection (ie. the solution to the original equation.)

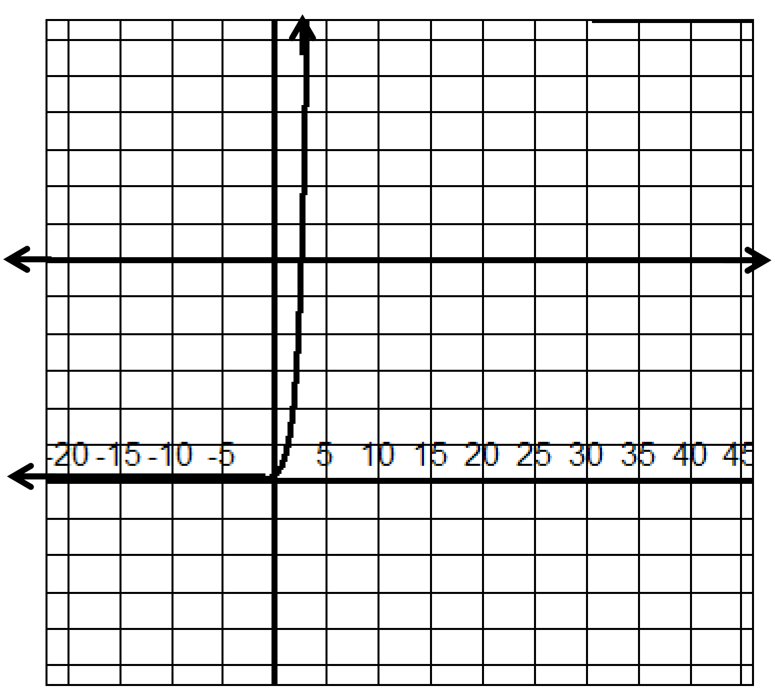
Example: Solve $4^x = 30$

<u>Algebraically we get:</u>

$$\log_4 4^x = \log_4 30$$
$$x = \log_4 30$$
$$x = \frac{\log 30}{\log 4} \approx 2.453$$

<u>On your calculator, graph</u>: $y_1=4^x$
$y_2=30$

At the point of intersection, $y_1= y_2$, and so also $4^x=30$. The x-coordinate of that point will be the solution to our equation.



Adjust window to
$-10 < x < 10$
$0 < y < 32$

Use [CALC] [intersect] to find the point of intersection. The point of intersection is (2.45, 30). Thus, the solution is x = 2.45.

*In General, to use the calculator to solve an equation graphically:

1. Graph the left and right sides of the equation separately.
2. Use the [CALC] [intersect] feature to find the point of intersection.

    - Press [CALC] [intersect].
    - Use the left and right arrows to move the cursor along the first curve to get as close to the point of intersection as possible. Press [ENTER].
    - Use the left and right arrows to move the cursor along the second curve to get as close to the point of intersection as possible. Press [ENTER] again.
    - Press [ENTER] to have the calculator make its guess. The coordinates of the point of intersection will appear at the bottom of the screen.

    Note: It does not matter which equation you start with. Pressing the up and down arrows will make the cursor jump from one graphed equations to the other.

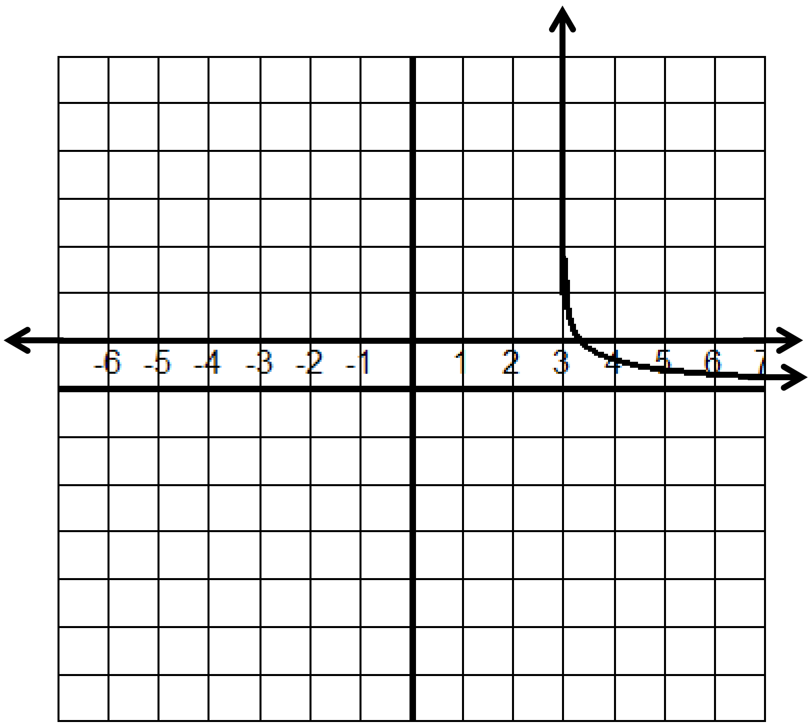
3. The x-coordinate of the point of intersection will be the solution to the original equation.

Example: Find the solution using a graphing calculator.

$$\log x - \log(x-3) = 1$$

Graph $y_1 = \log x - \log(x-3)$
$y_2 = 1$

Find the point of intersection:



Zoom in if necessary.

Use [CALC] [intersect] to find the point of intersection. The point of intersection is (3.33, 1). Thus, the solution is x ≈ 3.33.

The solution we found earlier by doing this algebraically was x = 10/3 ≈ 3.33.

Applications:

(1) How long would it take for an investment to double if the interest were compounded continuously at 8%?

Remember the formula for continuously compounding interest:

$$A = Pe^{rt}$$

If we want our principal to double, we must replace A with $2P$. We get:

$$2P = Pe^{0.08t}$$

Now solve for t. Start by dividing both sides by P.

$$2 = e^{0.08t}$$

$$\ln 2 = \ln e^{0.08t}$$

$$2 = 0.08t$$

$$t = \frac{2}{0.08} \approx 8.66 \text{ years}$$

(2) You have $50,000 to invest. You need to have $350,000 to retire in thirty years. At what continuously compounded interest rate would you need to invest to reach your goal?

$$A = Pe^{rt}$$

We need 350,000 = 50,000$e^{r(30)}$

Solve for r.

$$350{,}000 = 50{,}000e^{30r}$$

$$7 = e^{30r}$$

$$\ln 7 = \ln e^{30r}$$

$$\ln 7 = 30r$$

$$r = \frac{\ln 7}{30} \approx 0.065 = 6.5\%$$