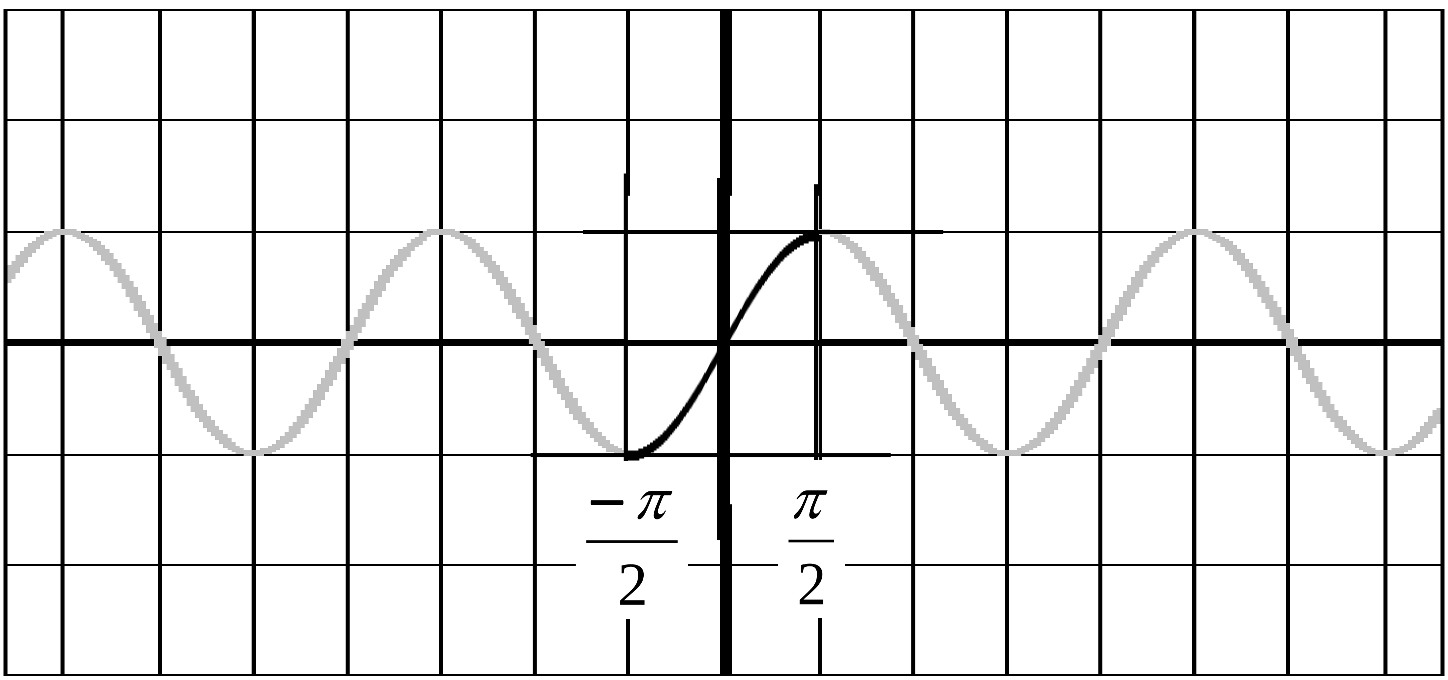
Inverse Trig Functions

Because the sine function does not pass the Horizontal Line Test, we must restrict its domain in order for its inverse to be a function. We restrict the domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

This is not one cycle of sine. It is a part of the cycle in which we have no y's repeating, so that it passes the Horizontal Line Test (ie. is one-to-one), and thus will have an inverse.



**Think of only using the values on the unit circle in the 1st and 4th quadrants, with all angle measures written as angles between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$.

Remember that the inverse of a function is found by interchanging x and y and then solve for y.

Thus, for $y = \sin x$, the inverse is $x = \sin y$.

We cannot solve for y, so we define the function for this inverse as $y = \arcsin x$.

Definition: The inverse sine function can be denoted by

$y = \arcsin x$ if and only if $x = \sin y$

where $-1 \le x \le 1$ and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$.

It can be thought of as the angle whose sine is x.

Note: $y = \arcsin x$ can also be written $y = \sin^{-1} x$.
(This is not the reciprocal of $\sin x$.)

Examples: Find the exact values of the following.

a) $\sin^{-1}(-1)$ answer: $\frac{-\pi}{2}$ (not $\frac{3\pi}{2}$)

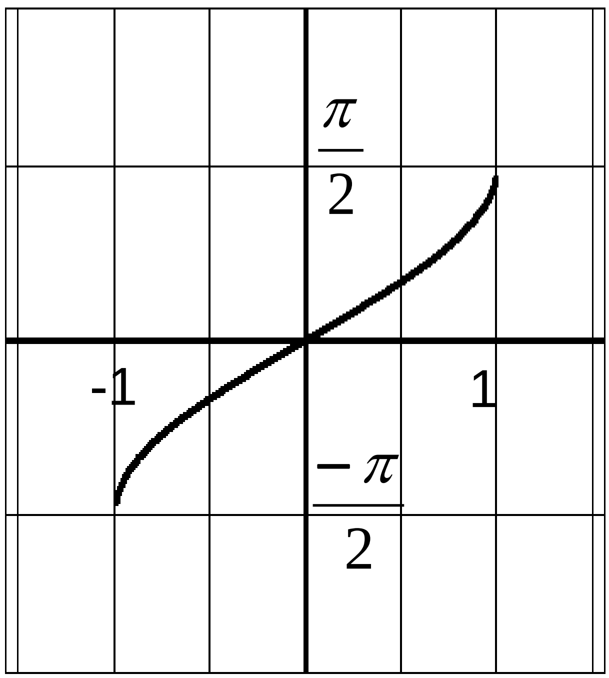
b) $\arcsin\left(\frac{1}{2}\right)$ answer: $\frac{\pi}{6}$

c) $\arcsin(2)$ answer: There is no angle whose sine is 2.

Graph *y* = arcsin *x.*

Make a table. Remember that $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$.

y	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x	-1	$\frac{-\sqrt{2}}{2}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

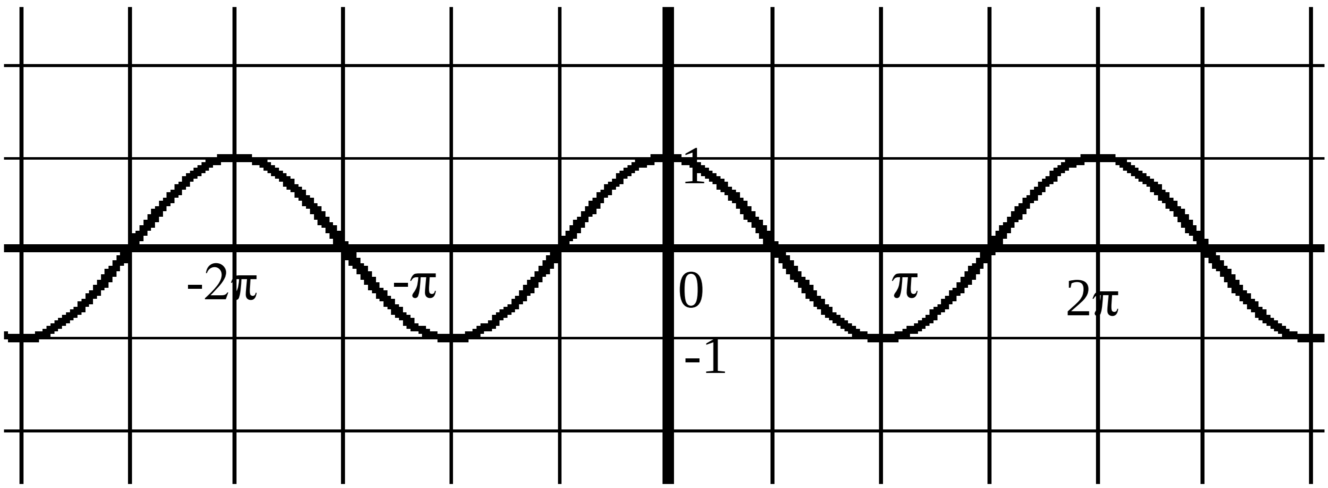


This graph is a reflection of the sine curve over the line y = x.

**Remember your restrictions on the domain and range for y = arcsin x.

Other Inverse Trig Functions

Look at the graph of y = cos x.



The section of the cosine graph that we will consider for arccosine is from 0 to π.

Definition: The inverse cosine function can be denoted by

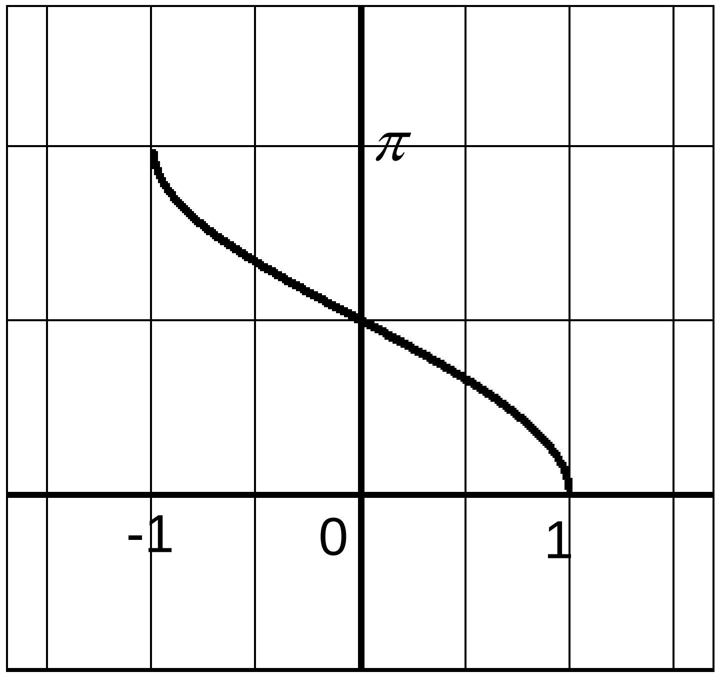
y = arccos x if and only if $x = \cos y$

where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.

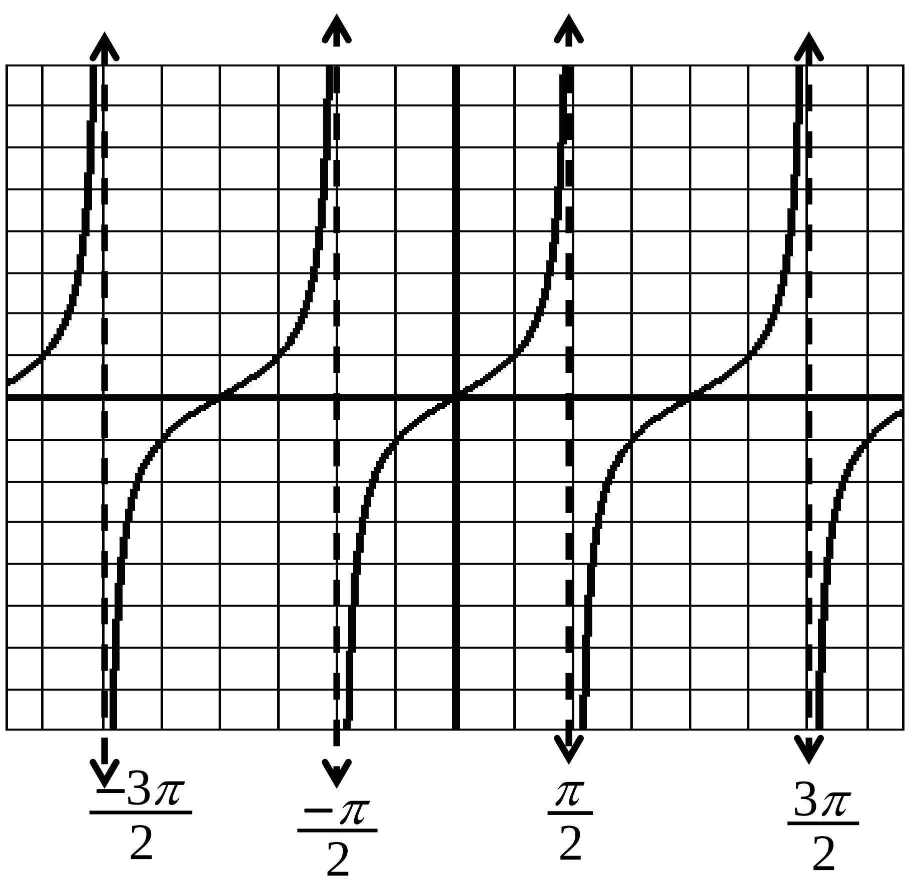
It can be thought of as the angle whose cosine is x.

Note: y = arccos x can also be written $y = \cos^{-1} x$.
(This is not the reciprocal of cos x.)

Graph y = arcos x.



Look at y = tan x.



The section of the tangent graph that we will consider for arctangent is from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$.

Definition: The <u>inverse tangent function</u> can be denoted by

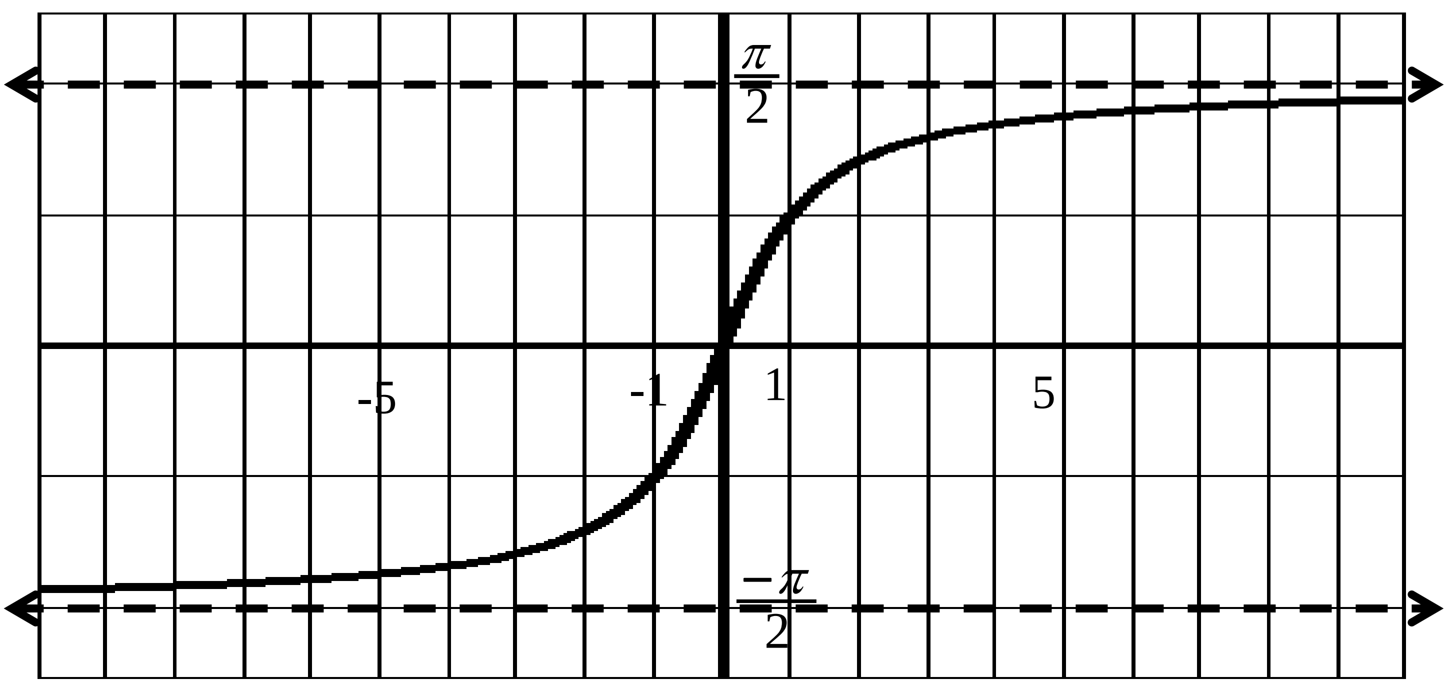
y = arctan x if and only if x = tan y

where x is a real number and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$.

It can be thought of as the angle whose tangent is x.

<u>Note</u>: y = arctan x can also be written $y = \tan^{-1} x$.
(This is not the reciprocal of tan x.)

Graph y = arctan x.



In General,

Function	*Domain*	*Range*
$y = \arcsin x \leftrightarrow \sin y = x$	[-1, 1]	$[-\pi/2, \pi/2]$
$y = \arccos x \leftrightarrow \cos y = x$	[-1, 1]	$[0, \pi]$
$y = \arctan x \leftrightarrow \tan y = x$	$[-\infty, \infty]$	$(-\pi/2, \pi/2)$

Example: Evaluate the following.

a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ answer: $\frac{\pi}{6}$

b) arctan -1 answer: $\frac{-\pi}{4}$ (not $\frac{3\pi}{4}$)

c) arcos $\left(\frac{-1}{2}\right)$ answer: $\frac{2\pi}{3}$

d) $\tan^{-1} 0$ answer: 0

e) arcsin $\left(\frac{-\sqrt{2}}{2}\right)$ answer: $\frac{-\pi}{4}$

**When finding these values, you must remember your ranges of the inverse functions!

Using a Calculator

For inverse trig functions, use the [2^{nd}] button along with the trig function.

Example: Use a calculator to evaluate the following in radians. (Set your calculator mode to radians.)

a) $\sin^{-1} 0.5524$ answer: [2nd] [sin] 0.5524 [ENTER]
≈ 0.5852

b) $\tan^{-1} -3.254$ answer: [2nd] [tan] -3.254 [ENTER]
≈ -1.2726

c) arcos 0.2345 answer: [2nd] [cos] 0.2345 [ENTER]
≈ 1.3341

Composition of Functions

Remember: $f(f^{-1}(x)) = x$ *and* $f^{-1}(f(x)) = x$.

Functions and their inverses “undo” each other.

So, for our inverse trig functions we have:

Inverse Properties of Trigonometric Functions

If -1 ≤ α ≤ 1 and $\frac{-\pi}{2}$.≤ β ≤ $\frac{\pi}{2}$., then

sin(arcsin α) = α and arcsin(sin β) = β.

If -1 ≤ α ≤ 1 and 0 ≤ β ≤ π, then

cos(arccos α) = α and arccos(cos β) = β.

If -∞ ≤ α ≤ ∞ and $\frac{-\pi}{2}$.≤ β ≤ $\frac{\pi}{2}$., then

tan(arctan α) = α and arctan(tan β) = β.

***Note: These properties only apply for the intervals given.

Example:

arcsin(sin $\frac{3\pi}{2}$) ≠ $\frac{3\pi}{2}$ because the arcsine must be between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$.

arcsin(sin $\frac{3\pi}{2}$) = arcsin (-1) = $\frac{-\pi}{2}$

Example: Find the exact value if possible.

a) sin(arcsin 0.12) answer: 0.12

b) arctan(tan $\frac{5\pi}{6}$) answer: $\frac{-\pi}{6}$

<u>Explanation</u>:

$\frac{5\pi}{6}$ does not lie in our range for arctangent. Consider the other 3 reference angles for $\frac{5\pi}{6}$ on the unit circle. They are $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. The only ones that fall in our range for arctangent are $\frac{\pi}{6}$ and $\frac{11\pi}{6}$ (***if*** we write it as $\frac{-\pi}{6}$). According to allsintancos, the tangent of $\frac{5\pi}{6}$ is negative. The only other angle that is in our range for arctangent <u>and</u> has a negative tangent would be $\frac{-\pi}{6}$.

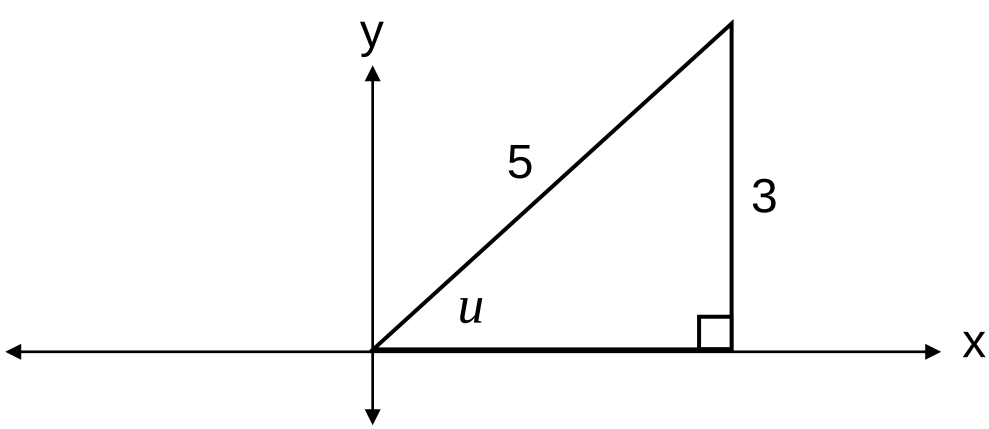
c) cos(arcos 6) *answer*: There is no angle whose cosine is 6.

Sometimes we must use right triangles to help find values.

Examples: Find the exact value.

a) $\cos(\sin^{-1} \frac{3}{5})$

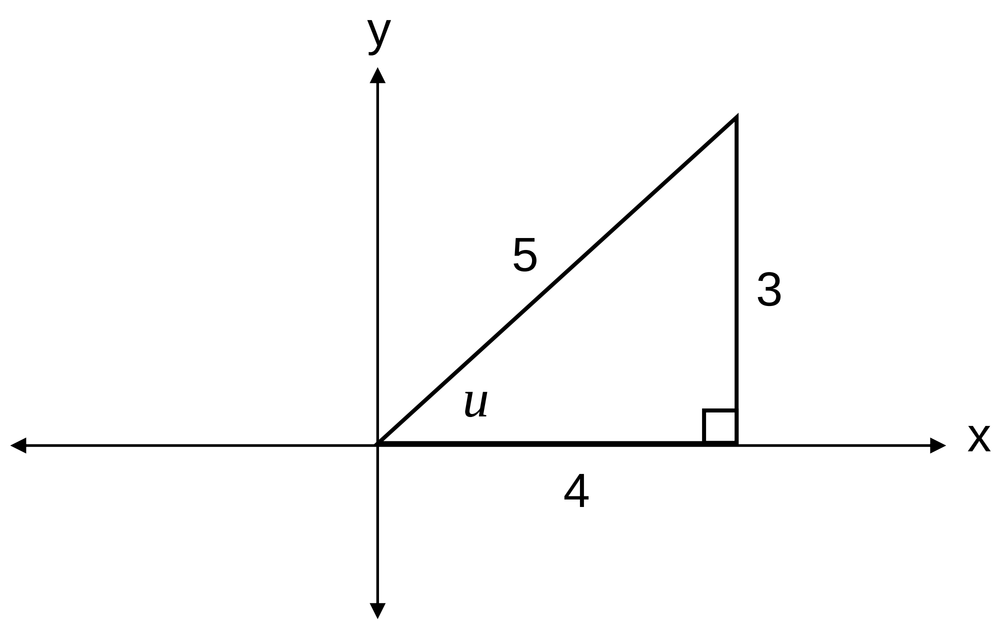
Now $\sin^{-1} \frac{3}{5}$ is the angle whose sine is $\frac{3}{5}$. Since SOHCAHTOA tells us sine = $\frac{opp}{hyp}$, let's draw a right triangle, using u as our unknown angle. We will let the opposite side = 3 and the hypotenuse = 5. Because the sine is a positive $\frac{3}{5}$, our angle must be in Quadrant 1.



We want the cosine of this angle, which is $\frac{adj}{hyp}$. But we don't know what the adjacent side is so we use the Pythagorean Theorem to find it.

$$adj^2 = 5^2 - 3^2$$
$$adj^2 = 16, \text{so } adj = 4$$

So we have

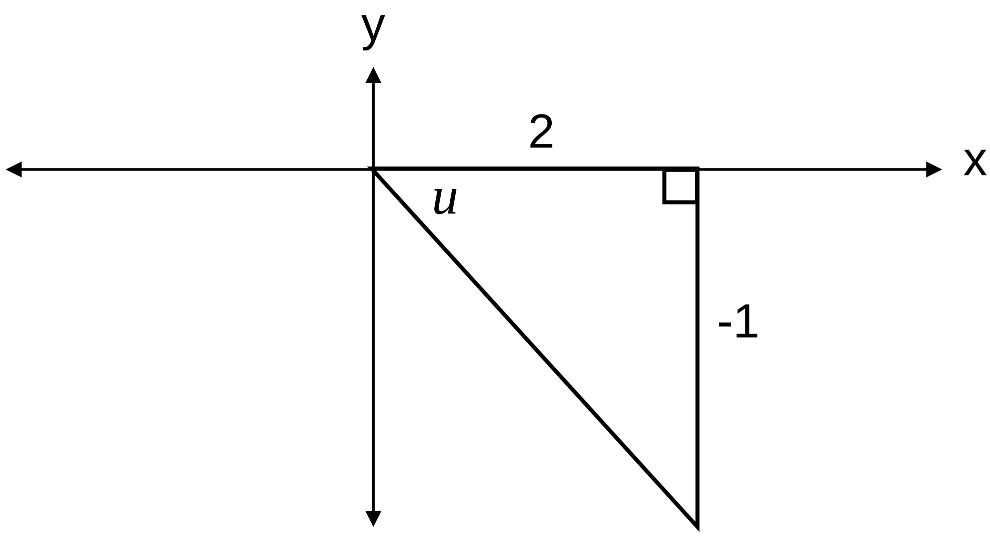


We can see that the cosine is $\frac{4}{5}$. Thus $\cos(\sin^{-1}\frac{3}{5}) = \frac{4}{5}$.

b) $\sin[\tan^{-1}(\frac{-1}{2})]$

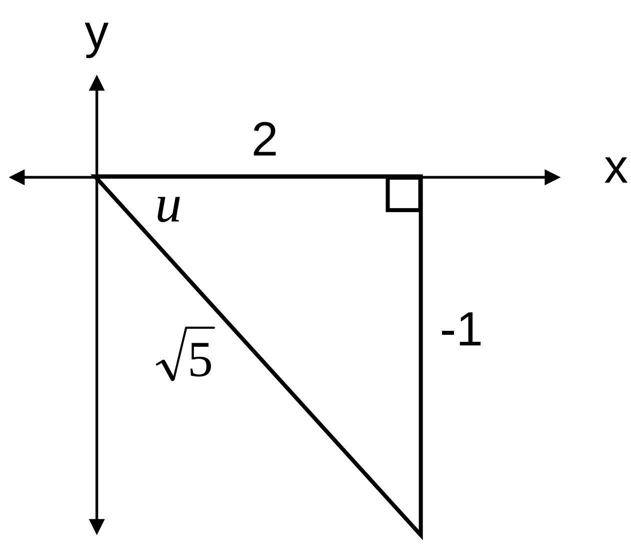
$\tan^{-1}(\frac{-1}{2})$] is an angle whose tangent is $\frac{-1}{2}$. Since our value for $\tan^{-1}(\frac{-1}{2})$ must be between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, then our angle must be in Quadrant 4 by allsintancos. Let's draw a triangle using opp = 1 and adj = 2.

**We will label the opposite side -1 to remind us that the y-coordinate in the 4th quadrant would be negative.



Use Pythagorean Theorem to find the hypotenuse.

$1^2 + 2^2 = hyp^2$
$hyp^2 = 5$
hyp = $\sqrt{5}$

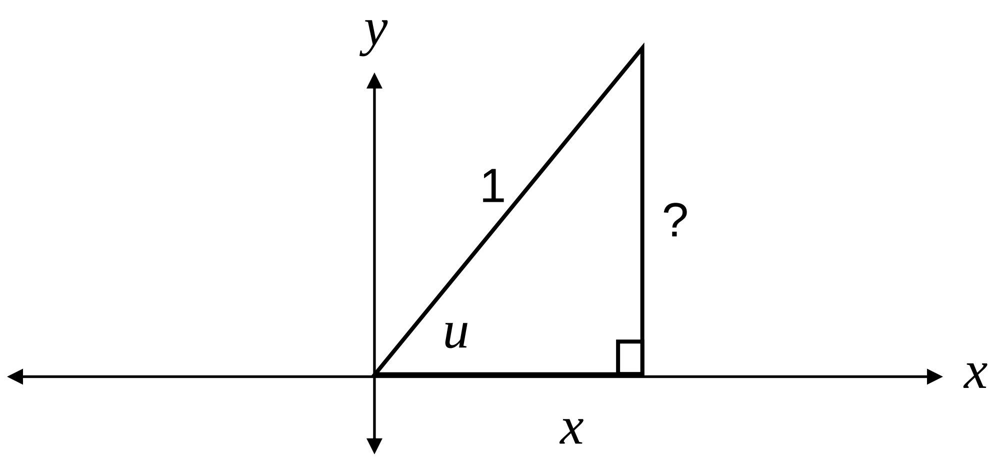


Now we can see that the sin of this angle is $\frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$.

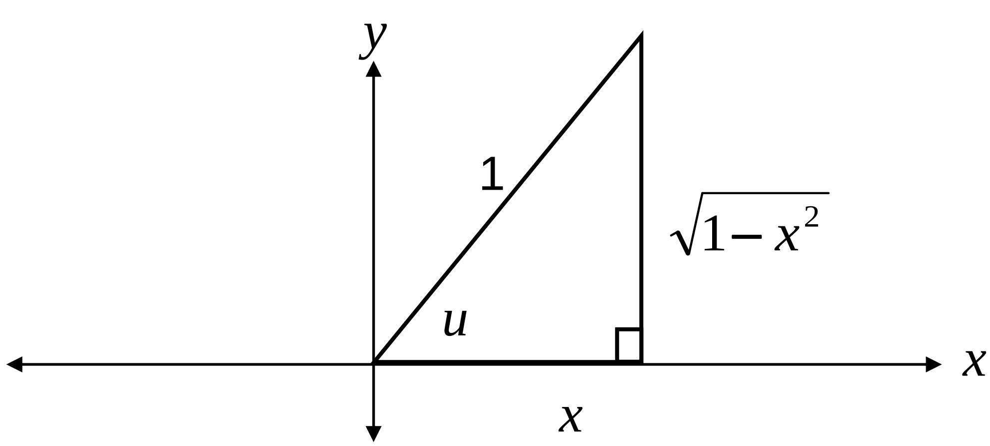
Thus, our answer is $\sin[\tan^{-1}(\frac{-1}{2})] = \frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$.

c) sin(arcos x) $0 \leq x \leq 1$

arcos x is the angle whose cosine is x. Since cosine is $\frac{adj}{hyp}$, let's assign x to the adjacent and let the hyp = 1.



By using Pythagorean's Theorem, we get that the side opposite our angle is $\sqrt{1^2 - x^2} = \sqrt{1 - x^2}$



Then the sine of this angle is $\frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.