Verifying Trigonometric Identities

Remember that an <u>identity</u> is an equation that is true for ***all*** real values on the domain of the variable.

Solving equations is different than verifying and identity. When we verify an identity, we are trying to work with one side to transform it into the other side.

Example: Verify the identity.

$$\frac{\sin^2 x - 1}{\sin^2 x} = -\cot^2 x$$

<u>Solution:</u>

Decide which side to work with. Then start with an equation that sets that side equal to itself. Then work with one of the sides until you attain what you are trying to verify.

$$\begin{aligned}\frac{\sin^2 x - 1}{\sin^2 x} &= \frac{\sin^2 x - 1}{\sin^2 x} \\ &= \frac{(1-\cos^2 x) - 1}{\sin^2 x} \\ &= \frac{-\cos^2 x}{\sin^2 x} \\ &= -\cot^2 x\end{aligned}$$

Alternate Solution:

$$\frac{\sin^2 x - 1}{\sin^2 x} = \frac{\sin^2 x - 1}{\sin^2 x}$$

$$= \frac{\sin^2 x}{\sin^2 x} - \frac{1}{\sin^2 x}$$

$$= 1 - \csc^2 x$$

$$= 1 - (1 + \cot^2 x)$$

$$= -\cot^2 x$$

Note: There is often more than one way to verify a trigonometric identity.

Example: Verify the identity.

$$\frac{1}{\sec x - 1} - \frac{1}{\sec x + 1} = 2\cot^2 x$$

We usually start with the more complicated side, so we will work with the left. Start by getting a common denominator and adding the fractions.

$$\frac{1}{\sec x - 1} - \frac{1}{\sec x + 1} = \frac{1}{\sec x - 1} - \frac{1}{\sec x + 1}$$

$$= \frac{1}{(\sec x - 1)} \cdot \frac{(\sec x + 1)}{(\sec x + 1)} - \frac{1}{(\sec x + 1)} \cdot \frac{(\sec x - 1)}{(\sec x - 1)}$$

$$= \frac{(\sec x + 1) - (\sec x - 1)}{(\sec x - 1)(\sec x + 1)}$$

$$= \frac{(\sec x + 1) - (\sec x - 1)}{(\sec^2 x - 1)}$$

$$= \frac{2}{\tan^2 x}$$

$$= 2 \cdot \frac{1}{\tan^2 x}$$

$$= 2\cot^2 x$$

Example: Verify the identity.

$$(1+\cot^2 x)(1-\sin^2 x) = \cot^2 x$$

Work with the left side. Notice that you have some Pythagorean identities.

$$\begin{aligned}(1+\cot^2 x)(1-\sin^2 x) &= (1+\cot^2 x)(1-\sin^2 x)\\ &= \csc^2 x\cos^2 x\\ &= \left(\frac{1}{\sin^2 x}\right)\cos^2 x\\ &= \frac{\cos^2 x}{\sin^2 x}\\ &= \cot^2 x\end{aligned}$$

Example: Verify the identity.

$$\sec u + \tan u = \frac{1}{\sec u - \tan u}$$

If you don't know what else to do, convert all terms to sine and cosine. Work with the left side.

$$\begin{aligned}
\sec u + \tan u &= \sec u + \tan u \\
&= \frac{1}{\cos u} + \frac{\sin u}{\cos u} \\
&= \frac{1+\sin u}{\cos u}
\end{aligned}$$

Notice that the numerator is close to the Pythagorean Identity. See if you can make it look like it.

$$\begin{aligned}
&= \frac{1+\sin u}{\cos u} \cdot \frac{(1-\sin u)}{(1-\sin u)} \\
&= \frac{1-\sin^2 u}{\cos u(1-\sin u)} \\
&= \frac{\cos^2 u}{\cos u(1-\sin u)} \\
&= \frac{\cos u}{(1-\sin u)}
\end{aligned}$$

Now look at what you are trying to verify. You need a 1 in the numerator and now have cos u. To make cos u = 1, you need to multiply it by $\frac{1}{\cos u}$. This is the same as sec u.

$$= \frac{\cos u}{(1 - \sin u)} \cdot \frac{(\sec u)}{(\sec u)}$$

$$= \frac{1}{\sec u - \sin u \sec u}$$

$$= \frac{1}{\sec u - \sin u \dfrac{1}{\cos u}}$$

$$= \frac{1}{\sec u - \dfrac{\sin u}{\cos u}}$$

$$= \frac{1}{\sec u - \tan u}$$

Sometimes you need to look at what you are trying to get to decide on the next step.

Example: Verify the identity.

$$\cot t \cos t = \csc t - \sin t$$

Take the right side and try writing it in terms of sine and cosine.

$$\begin{aligned}
\csc t - \sin t &= \csc t - \sin t \\
&= \frac{1}{\sin t} - \sin t \\
&= \frac{1}{\sin t} - \sin t \cdot \frac{\sin t}{\sin t} \\
&= \frac{1 - \sin^2 t}{\sin t} \\
&= \frac{\cos^2 t}{\sin t} \\
&= \frac{\cos t}{\sin t} \cdot \cos t \\
&= \cot t \cos t
\end{aligned}$$

Example: Verify the identity.

$$\frac{\cot^2 u}{1+\csc u} = \frac{1-\sin u}{\sin u}$$

Start with the left side.

$$\begin{aligned}\frac{\cot^2 u}{1+\csc u} &= \frac{\cot^2 u}{1+\csc u} \\ &= \frac{\csc^2 u - 1}{1+\csc u} \\ &= \frac{(\csc u + 1)(\csc u - 1)}{1+\csc u} \\ &= \csc u - 1\end{aligned}$$

We seem to be stuck. Start over and work with the right side.

$$\begin{aligned}\frac{1-\sin u}{\sin u} &= \frac{1-\sin u}{\sin u} \\ &= \frac{1}{\sin u} - \frac{\sin u}{\sin u} \\ &= \csc u - 1\end{aligned}$$

Notice that we got to the same place with both sides.

Start over and put the 2 parts together.

$$\begin{aligned}\frac{\cot^2 u}{1+\csc u} &= \frac{\cot^2 u}{1+\csc u}\\ &= \frac{\csc^2 u - 1}{1+\csc u}\\ &= \frac{(\csc u + 1)(\csc u - 1)}{1+\csc u}\\ &= \csc u - 1\\ &= \frac{1}{\sin u} - \frac{\sin u}{\sin u}\\ &= \frac{1-\sin u}{\sin u}\end{aligned}$$

Sometimes it is helpful to work with both sides until you get something that matches. Then write the verification working with only one side, but using the information you gained from working the other side, too.

There is no well-defined set of rules to follow in verifying trig identities. Practice is your best weapon. Here are some guidelines that are helpful:

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.

2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.

3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.

4. If the preceding guidelines do not help, try converting all terms to sines and cosines.

5. Always try something. Even paths that lead to dead ends give you insights.