Solving Trigonometric Equations

The preliminary goal in solving a trig equation is to isolate the trig function first.

Example: Solve $1-2\cos x=0$.

Isolate the cos x term like you would isolate any variable term in an algebraic equation.

$$1-2\cos x=0$$

$$-2\cos x=-1$$

$$\cos x=\frac{1}{2}$$

We know that if our angle is from 0 to 2π, we have

$$x=\frac{\pi}{3} \text{ or } x=\frac{5\pi}{3}$$

But there are many angles that are coterminal with these 2 angles. Every time we add a multiple of 2π we get a coterminal angle. Thus, the solution should be

$$x=\frac{\pi}{3}+2n\pi \text{ or } x=\frac{5\pi}{3}+2n\pi$$

Example: Solve $\sin x + 1 = -\sin x$.

$$\sin x + 1 = -\sin x$$
$$2\sin x = -1$$
$$\sin x = \frac{-1}{2}$$

Since sine has a period of 2π, find the solutions on the interval from 0 to 2π first (i.e. on the interval $[0, 2\pi)$).

$$x = \frac{7\pi}{6} \text{ or } x = \frac{11\pi}{6}$$

Every time we add a multiple of 2π to these solutions, we get another solution. Thus, we write the solution as

$$x = \frac{7\pi}{6} + 2n\pi \text{ or } x = \frac{11\pi}{6} + 2n\pi$$

Example: Solve $\tan^2 x - 3 = 0$.

$$\tan^2 x = 3$$
$$\sqrt{\tan^2 x} = \pm\sqrt{3}$$
$$\tan x = \pm\sqrt{3}$$

We had to isolate the trig function and then take the square root of both sides.

Because the period of tangent is π, we will first find all the solutions on the interval from 0 to π. We write this interval as [0, π). Our solution is then

$$x = \frac{\pi}{3} \text{ or } x = \frac{2\pi}{3}$$

Every time we add a multiple of π to either of these solutions we get an angle that has the same tangent. Thus, our solution is

$$x = \frac{\pi}{3} + n\pi \text{ or } x = \frac{2\pi}{3} + n\pi$$

Checking a Solution Graphically

For the equation $\tan^2 x - 3 = 0$ we got the solution

$$x = \frac{\pi}{3} + n\pi \text{ or } x = \frac{2\pi}{3} + n\pi$$

When solving any equation graphically, we graph the left side and right side separately, and then see where the 2 graphs intersect.

If the equation is written to =0, we can graph it on the calculator and see where it crosses the x-axis to find the solutions. This can be done by using the [CALC] [zero] feature or the [TRACE] feature.

Graph $y = \tan^2 x - 3$.

Use the [CALC] [zero] feature to find the approximate values of the x-intercepts. These are the solutions of the equation.

Or, use [TRACE] to find the approximate values of the x-intercepts.

Example: Solve $\sec x \csc x = \csc x$.

Do not divide both sides by csc x. This would eliminate possible solutions. Instead, when the equation involves more than one trig function, collect all terms on one side and try to separate the functions by factoring or using identities.

$$\sec x \csc x = \csc x$$
$$\sec x \csc x - \csc x = 0$$
$$\csc x(\sec x - 1) = 0$$
$$\csc x = 0 \quad \text{or} \quad \sec x - 1 = 0$$
$$\csc x = 0 \quad \text{or} \quad \sec x = 1$$

There is no angle whose cosecant is 0, since csc x = 0 means $\sin x = \frac{1}{0}$, which is undefined (and never happens for the sine function). So we look at the other part.

If $sec\ x$ = 1, then $\cos x = \frac{1}{1} = 1$.

This happens at 0, 2π, 4π, etc. We write the solution as

$$x = 2n\pi$$

Equations of Quadratic Type

Example: Solve $2\cos^2 x + \cos x - 1 = 0$
on the interval $[0, 2\pi)$.

(Think of this as $2u^2 + u - 1 = 0$.)

$$2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

$$2\cos x - 1 = 0 \text{ or } \cos x + 1 = 0$$

$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}, \text{or } \pi$$

Example: Solve $2\cos^2 x + 3\sin x - 3 = 0$ on the interval $[0, 2\pi)$.

Before attempting this problem, we must write it as a single trig function. To do this, remember that we have the Pythagorean identity that says $\sin^2 x + \cos^2 x = 1$ which can also be written as $\cos^2 x = 1 - \sin^2 x$. Substitute in the above equation for $\cos^2 x$.

$$2\cos^2 x + 3\sin x - 3 = 0$$

$$2(1 - \sin^2 x) + 3\sin x - 3 = 0$$

$$2 - 2\sin^2 x + 3\sin x - 3 = 0$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$(2\sin x - 1)(\sin x - 1) = 0$$

$$2\sin x - 1 = 0 \text{ or } \sin x - 1 = 0$$

$$\sin x = \frac{1}{2} \text{ or } \sin x = 1$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{\pi}{2}$$

Example: Solve $\sec x + 1 = \tan x$.

Secant and tangent are related, but only if you have $\tan^2 x$ or $\sec^2 x$. If we square both sides of the equation, we will have a $\tan^2 x$ on the right.

$$\sec x + 1 = \tan x$$
$$(\sec x + 1)^2 = (\tan x)^2$$
$$\sec^2 x + 2\sec x + 1 = \tan^2 x$$

Now substitute the Pythagorean identity $\tan^2 x = \sec^2 x - 1$.

$$\sec^2 x + 2\sec x + 1 = \tan^2 x$$
$$\sec^2 x + 2\sec x + 1 = \sec^2 x - 1$$
$$2\sec x + 1 = -1$$
$$2\sec x = -2$$
$$\sec x = -1$$

We know that if sec x = -1, then cos x = $\frac{1}{-1}$ =-1. Thus,

$$x = \pi + 2n\pi$$

*Remember to check your solution, because squaring both sides sometimes introduces extraneous solutions!!

Funtions Involving Multiple Angles

If the argument of the trig function is a multiple angle of the form $\sin ku$ and $\cos ku$, then first solve the equation for ku and then solve for u.

Example: Solve $2\sin 2t + 1 = 0$ on the interval [0, 2π).

$$2\sin 2t + 1 = 0$$
$$2\sin 2t = -1$$
$$\sin 2t = \frac{-1}{2}$$

Ask the question, "The sine of what angle is $\frac{-1}{2}$?"

The answer is $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$. So $2t = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

Since 0 ≤t ≤ 2π, then 0 ≤2t ≤ 4π. This means we are actually looking for answers from 0 to 4π.

So we have $2t = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$.

Divide thru by 2 to get $t = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

Example: Solve the equation $\cot\frac{x}{2}+1=0$ on the interval $[0,2\pi)$.

$$\cot\frac{x}{2}+1=0$$

$$\cot\frac{x}{2}=-1$$

The cotangent = -1 at $\frac{3\pi}{4}$ so $\frac{x}{2}$ must = $\frac{3\pi}{4}$.

Take $\frac{x}{2}=\frac{3\pi}{4}$ and solve for x.

$$\frac{x}{2}=\frac{3\pi}{4}$$

$$(2)\cdot\frac{x}{2}=\frac{3\pi}{4}\cdot(2)$$

$$x=\frac{3\pi}{2}$$

Using Inverse Functions

Example: Solve $\sec^2 x - 3\sec x - 10 = 0$ for all values of x.

$$\sec^2 x - 3\sec x - 10 = 0$$
$$(\sec x - 5)(\sec x + 2) = 0$$
$$\sec x - 5 = 0 \text{ or } \sec x + 2 = 0$$
$$\sec x = 5 \text{ or } \sec x = -2$$

We can solve these equations by taking the $\sec^{-1}$ of both sides, but we need to remember that the angle returned for arcsecant is between 0 and π (the range of arcsecant).

For sec x = 5, the angle x must be in Quadrant I and IV. Taking the $\sec^{-1}$ of both sides will give us the Quadrant I angle. The angle in Quadrant IV will be the opposite of that angle.

$$\sec x = 5$$
$$\sec^{-1}(\sec x) = \sec^{-1}(5)$$
$$x = \sec^{-1}(5)$$

The Quadrant I angle is $x = \sec^{-1} 5$.
The Quadrant IV angle is $x = -\sec^{-1} 5$.

For sec x = -2, the angle x must be in Quadrant II and III. Taking the $\sec^{-1}$ of both sides will give us the Quadrant II angle. The angle in Quadrant III will be the opposite of the Quadrant II angle.

$$\sec x = -2$$
$$\sec^{-1}(\sec x) = \sec^{-1}(-2)$$
$$x = \sec^{-1}(-2)$$

The Quadrant II angle is $x = \sec^{-1}(-2) = \cos^{-1}(-1/2) = \frac{2\pi}{3}$.

The Quadrant III angle is $x = -\sec^{-1}(-2) = \frac{-2\pi}{3}$.

Since we were asked to find all angles that are solutions, we must add $2n\pi$ to each angle. Thus, our solution is

$$x = \sec^{-1}(5) + 2n\pi$$
$$x = -\sec^{-1}(5) + 2n\pi$$
$$x = \frac{2\pi}{3} + 2n\pi$$
$$x = \frac{-2\pi}{3} + 2n\pi$$