Sum and Difference Formulas

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$$\sin(u+v)=\sin u\cos v+\cos u\sin v$$
$$\sin(u-v)=\sin u\cos v-\cos u\sin v$$
$$\cos(u+v)=\cos u\cos v-\sin u\sin v$$
$$\cos(u-v)=\cos u\cos v+\sin u\sin v$$
$$\tan(u+v)=\frac{\tan u+\tan v}{1-\tan u\tan v}$$
$$\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$$

These formulas help us to find exact values of trig functions involving sums and differences of special angles from our unit circle.

Example: Find the exact value of sin 15°.

sin 15°= sin(45°- 30°)

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Example: Find the exact value of $\cos \frac{7\pi}{12}$.

$$\begin{aligned}
\cos\frac{7\pi}{12} &= \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right) \\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

Example: Find the exact value of cos 105°.

$$\begin{aligned}
\cos 105° &= \cos(45° + 60°) \\
&= \cos 45° \cos 60° - \sin 45° \sin 60° \\
&= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

Example: Find the exact value of the following:

$$\cos 78° \cos 18° + \sin 78° \sin 18°$$

This looks like it came from

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

So we have

$$\cos 78° \cos 18° + \sin 78° \sin 18° = \cos(78° - 18°) = \cos 60° = \frac{1}{2}$$

Example: Find the exact value of

$\sin 10° \cos 35° + \cos 10° \sin 35°$

This looks like it came from

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

So we have

$$\sin 10° \cos 35° + \sin 10° \sin 35° = \sin(10° + 35°) = \sin 45° = \frac{\sqrt{2}}{2}$$

Example: Write $\tan(\tan^{-1}(-1)-\tan^{-1}x)$ as an algebraic expression.

This is the tangent difference formula.

Let $u = \tan^{-1}(-1)$
Let $v = \tan^{-1}x$

$$\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$$

$$\tan(\tan^{-1}(-1)-\tan^{-1}x)=\frac{\tan\left(\tan^{-1}(-1)\right)-\tan\left(\tan^{-1}x\right)}{1+\tan\left(\tan^{-1}(-1)\right)\tan\left(\tan^{-1}x\right)}$$

$$=\frac{-1-x}{1+(-1)x}$$

$$=\frac{-1-x}{1-x}$$

Example: Verify the cofunction identity sin(90°- x)=cos x.

$$\sin(90°-x)=\sin 90°\cos x-\cos 90°\sin x$$

$$=\quad 1\cdot\cos x-\quad 0\cdot\sin x$$

$$=\cos x$$

Example: Simplify sin(x+3π).

$$\begin{aligned}\sin(x+3\pi) &= \sin x\cos 3\pi + \cos x \sin 3\pi \\ &= (\sin x)(-1) + (\cos x)(0) \\ &= -\sin x\end{aligned}$$

Example: Verify the identity:

$$\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$$

$\sin(x + y)\sin(x - y)$

$= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$

**This is a difference of squares.*

$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$

**Substitute for the* $\sin^2 x$ *and* $\sin^2 y$*.*

$= (1 - \cos^2 x)\cos^2 y - \cos^2 x (1 - \cos^2 y)$

**Multiply using Distributive.*

$= \cos^2 y - \cos^2 x \cos^2 y - \cos^2 x + \cos^2 x \cos^2 y$

**Combine like terms.*

$= \cos^2 y - \cos^2 x$

Example: Solve the following in the interval [0, 2π).

$$\cos\left(x+\frac{\pi}{4}\right)-\cos\left(x-\frac{\pi}{4}\right)=1$$

Expand each cosine using the sum and difference formulas.

$$\cos\left(x+\frac{\pi}{4}\right)-\cos\left(x-\frac{\pi}{4}\right)=1$$

$$\left(\cos x\cos\frac{\pi}{4}-\sin x\sin\frac{\pi}{4}\right)-\left(\cos x\cos\frac{\pi}{4}+\sin x\sin\frac{\pi}{4}\right)=1$$

$$\cos x\cos\frac{\pi}{4}-\sin x\sin\frac{\pi}{4}-\cos x\cos\frac{\pi}{4}-\sin x\sin\frac{\pi}{4}=1$$

$$-2\sin x\sin\frac{\pi}{4}=1$$

$$-2\sin x\left(\frac{\sqrt{2}}{2}\right)=1$$

$$\left(-\sqrt{2}\right)\sin x=1$$

$$\sin x=\frac{-1}{\sqrt{2}}=\frac{-\sqrt{2}}{2}$$

$$x=\frac{5\pi}{4}\ \text{ or }\ \frac{7\pi}{4}$$