Multiple-Angle and Product-to-Sum Formulas

Sum and Difference Formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$
$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Double-Angle Formulas

Find $\sin 2u$.

Use the Sum Formula:

$$\begin{aligned} \sin 2u &= \sin(u+u) \\ &= \sin u \cos u + \cos u \sin u \\ &= 2 \sin u \cos u \end{aligned}$$

Find $\cos 2u$.

$$\begin{aligned}\cos 2u &= \cos(u+u)\\ &= \cos u \cos u - \sin u \sin u\\ &= \cos^2 u - \sin^2 u\end{aligned}$$

We can change the form of this using Pythagorean Identities:

$$\begin{aligned}\cos^2 u - \sin^2 u &= (1-\sin^2 u) - \sin^2 u\\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\begin{aligned}\cos^2 u - \sin^2 u &= \cos^2 u - (1-\cos^2 u)\\ &= 2\cos^2 u - 1\end{aligned}$$

After doing a similar derivation for $\tan 2u$, we get the following formulas:

<u>Double-Angle Formulas</u>

$$\sin 2u = 2\sin u \cos u$$

$$\tan 2u = \frac{2\tan u}{1-\tan^2 u}$$

$$\begin{aligned}\cos 2u &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$

Example: Solve $\sin 2x - \cos x = 0$ for $0 \le x \le 2\pi$.

$$\sin 2x - \cos x = 0$$
$$(2\sin x\cos x) - \cos x = 0$$
$$\cos x(2\sin x - 1) = 0$$
$$\cos x = 0 \text{ or } 2\sin x - 1 = 0$$
$$\cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

$\cos x = 0$ at $\frac{\pi}{2}, \frac{3\pi}{2}$ and $\sin x = \frac{1}{2}$ at $\frac{\pi}{6}, \frac{5\pi}{6}$

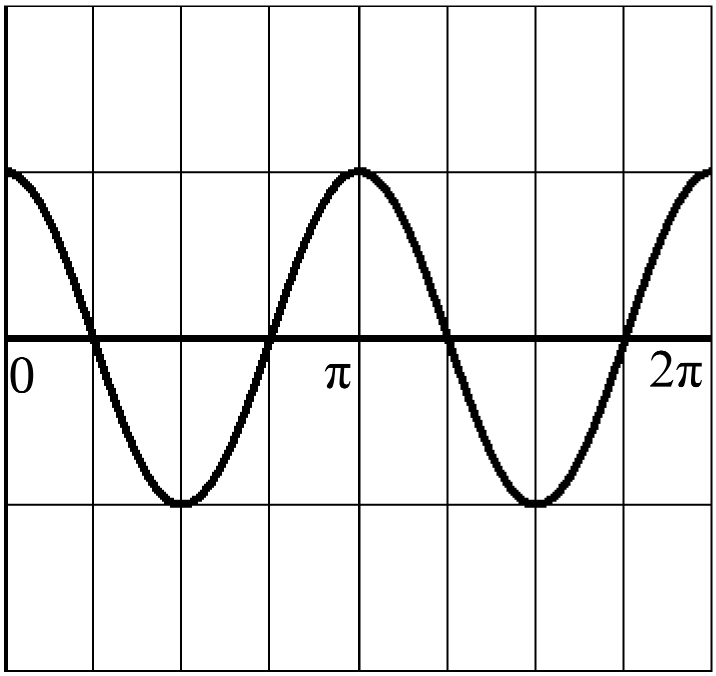
Solution: $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

Example: Sketch the graph of $y = \cos^4 x - \sin^4 x$ on $[0, 2\pi]$.

Factor the difference of squares:

$$y = \cos^4 x - \sin^4 x$$
$$= (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$$
$$= (\cos 2x) \cdot (1)$$
$$= \cos 2x$$

Now graph $y=\cos 2x$ for $0 \le x \le 2\pi$.



amplitude = 1

$0 \le 2x \le 2\pi$
$0 \le x \le \pi$

One period of cosine goes from 0 to π.

We need to graph from 0 to 2π, so we will graph 2 periods.

Example: Given $\sin x = \frac{12}{13}$ and $\frac{\pi}{2} \le x \le \pi$, find $\sin 2x$ $\cos 2x$ and $\tan 2x$.

Use the Pythagorean identity to find $\cos x$.

$$\cos^2 x = 1 - \frac{144}{169}$$

$$\cos^2 x = \frac{25}{169}$$

$$\cos x = \pm\frac{5}{13}$$

Since our angle is in quadrant II, we know that the cosine is negative, so

$$\cos x = \frac{-5}{13}$$

$$\sin 2x = 2\sin x\cos x = 2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) = \frac{-120}{169}$$

$$\cos 2x = 1 - 2\sin^2 x = 1 - 2\left(\frac{12}{13}\right)^2 = \frac{-119}{169}$$

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{\frac{-120}{169}}{\frac{-119}{169}} = \frac{-120}{169}\cdot\frac{169}{-119} = \frac{120}{119}$$

<u>Notice</u> that it was easier to use $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$ instead of the tan $2x$ formula, but you could do it either way.

From the double-angle formulas we get:

$$\sin 4u = 2\sin 2u\cos 2u$$
$$\sin 6u = 2\sin 3u\cos 3u$$
$$\sin 8u = 2\sin 4u\cos 4u$$
$$etc.$$

$$\cos 4u = \cos^2 2u - \sin^2 2u$$
$$\cos 6u = \cos^2 3u - \sin^2 3u$$
$$\cos 8u = \cos^2 4u - \sin^2 4u$$
$$etc.$$

Example: Derive a formula for $\sin 5x$.

$$\begin{aligned}
\sin 5x &= \sin(4x + x) \\
&= \sin 4x \cos x + \cos 4x \sin x \\
&= [\sin 2(2x)]\cos x + [\cos 2(2x)]\sin x \\
&= [2\sin 2x \cos 2x]\cos x + [1 - 2\sin^2 2x]\sin x \\
&= 2\sin 2x \cos 2x \cos x + \sin x - 2\sin x \sin^2 2x \\
&= 2\cos x[2\sin x \cos x][1 - 2\sin^2 x] - 2\sin x[2\sin x \cos x]^2 \\
&= 4\cos^2 x \sin x[1 - 2\sin^2 x] - 8\cos^2 x \sin^3 x \\
&= 4\cos^2 x \sin x - 8\cos^2 x \sin^3 x - 8\cos^2 x \sin^3 x \\
&= 4\cos^2 x \sin x - 16\cos^2 x \sin^3 x \\
&= 4\sin x(1 - \sin^2 x) - 16\sin^3 x(1 - \sin^2 x) \\
&= 4\sin x - 4\sin^3 x - 16\sin^3 x + 16\sin^5 x \\
&= 4\sin x - 20\sin^3 x + 16\sin^5 x
\end{aligned}$$

Power-Reducing Formulas

Look at $\cos 2u = 1 - 2\sin^2 u$ and solve for $\sin^2 u$.

$$\cos 2u = 1 - 2\sin^2 u$$

$$2\sin^2 u = 1 - \cos 2u$$

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

The double-angle formula can be used to obtain the following power-reducing formulas.

Power-Reducing Formulas

$$\sin^2 u = \frac{1-\cos 2u}{2} \qquad \cos^2 u = \frac{1+\cos 2u}{2} \qquad \tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Example: Rewrite $\cos^4 x$ as a sum of first powers of the cosine function.

$$\begin{aligned}
\cos^4 x = \left(\cos^2 x\right)^2 &= \left(\frac{1+\cos 2x}{2}\right)^2 \\
&= \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right) \\
&= \frac{1}{4}\left(1+2\cos 2x+\left[\frac{1+\cos 4x}{2}\right]\right) \\
&= \frac{1}{4}\left(1+2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right) \\
&= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x \\
&= \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x \\
&= \frac{1}{8}\left(3+4\cos 2x+\cos 4x\right)
\end{aligned}$$

Half-Angle Formulas

Look at the power-reduction for sine. Replace u with $\frac{u}{2}$.

$$\sin^2 \frac{u}{2} = \frac{1-\cos u}{2}$$

Note that the argument for cosine is twice the argument for sine^2. Now, take the square root of both sides:

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

This is a Half-Angle formula, and the sign depends on what quadrant $\frac{u}{2}$ is in.

Half-Angle Formulas

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos \frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan \frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\sin u}$$

The sign depends on the quadrant in which $\frac{u}{2}$ lies.

Example: Find the exact value of cos 165°.

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\cos 165° = \cos\frac{330°}{2} = \pm\sqrt{\frac{1+\cos 330°}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \pm\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}$$

$$= \pm\sqrt{\frac{2}{4}+\frac{\sqrt{3}}{4}}$$

$$= \pm\sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \pm\frac{\sqrt{2+\sqrt{3}}}{2}$$

Since 165° is in quadrant 2, the cosine is negative, and

$$\cos 165° = -\frac{\sqrt{2+\sqrt{3}}}{2}$$

Example: Solve $2\sin^2\frac{x}{2} = \cos x$ on $[0,2\pi)$.

If we use the half-angle formula for sine, we can get the equation all in terms of cosine.

$$2\sin^2\frac{x}{2} = \cos x$$

$$2\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2 = \cos x$$

$$2\left(\frac{1-\cos x}{2}\right) = \cos x$$

$$1-\cos x = \cos x$$

$$1 = 2\cos x$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Example: Solve $2-\sin^2 x = 2\cos^2 \frac{x}{2}$ on $[0,2\pi)$.

$$2-\sin^2 x = 2\cos^2 \frac{x}{2}$$

$$2-\sin^2 x = 2\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2$$

$$2-\sin^2 x = 2\left(\frac{1+\cos x}{2}\right)$$

$$2-\sin^2 x = 1+\cos x$$

$$2-\left(1-\cos^2 x\right) = 1+\cos x$$

$$1+\cos^2 x = 1+\cos x$$

$$\cos^2 x - \cos x = 0$$

$$\cos x(\cos x - 1) = 0$$

$$\cos x = 0 \quad or \quad \cos x = 1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, 0$$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$

$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$

$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

(These formulas can be verified using the sum and difference formulas.)

Example: Rewrite $\sin 2x \sin x$ as a difference.

Use the formula that is the product of sines.

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

$$\sin 2x \sin x = \frac{1}{2}\left[\cos(2x-x) - \cos(2x+x)\right]$$

$$= \frac{1}{2}\left[\cos x - \cos 3x\right]$$

$$= \frac{1}{2}\cos x - \frac{1}{2}\cos 3x$$

Sum-to-Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Example: Find the exact value of sin 195°- sin 105°.

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\sin 195^\circ - \sin 105^\circ = 2\cos\left(\frac{195^\circ + 105^\circ}{2}\right)\sin\left(\frac{195^\circ - 105^\circ}{2}\right)$$

$$= 2\cos\left(\frac{195^\circ + 105^\circ}{2}\right)\sin\left(\frac{195^\circ - 105^\circ}{2}\right)$$

$$= 2\cos\left(\frac{300^\circ}{2}\right)\sin\left(\frac{90^\circ}{2}\right)$$

$$= 2\cos 150^\circ \sin 45^\circ$$

$$= 2\left(\frac{-\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{-\sqrt{6}}{2}$$

Example : Solve $\cos 3x + \cos x = 0$ on $[0,2\pi)$.

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos 3x + \cos x = 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

So we have

$$2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 0$$
$$2\cos\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right) = 0$$
$$2\cos 2x \cos x = 0$$
$$\cos 2x = 0 \quad \text{or} \quad \cos x = 0$$

If $\cos 2x=0$, then $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \ldots$

which means that $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \ldots$

If $\cos x=0$, then $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \ldots$

Since we are restricted to the interval $[0, 2\pi)$, we have:

$$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$$

Example : Verify $\dfrac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = -\cot 6x$

$$\frac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = \frac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x}$$

$$= \frac{2\cos\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}{-2\sin\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}$$

$$= \frac{2\cos 6x \sin x}{-2\sin 6x \sin x}$$

$$= -\frac{\cos 6x}{\sin 6x}$$

$$= -\cot 6x$$