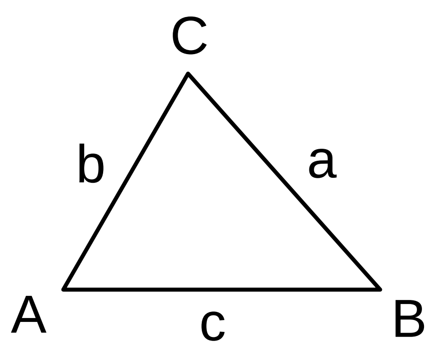
Law of Sines

We have learned how to use trigonometry to solve right triangles. Now we will look at how trigonometry can help us solve oblique triangles.

Definition: An oblique triangle is one that does not contain a right angle.



****To solve an oblique triangle, we need to know at least one side and any two other parts of the triangle.**

We get the following 4 cases:

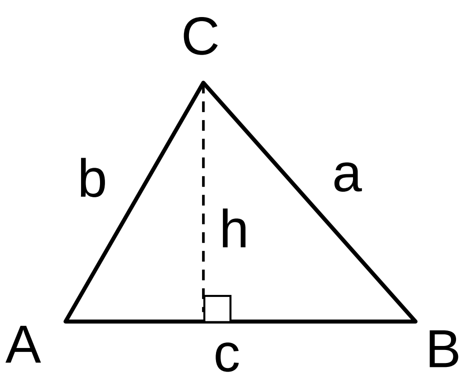
1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

We use the Law of Sines for the first two cases.

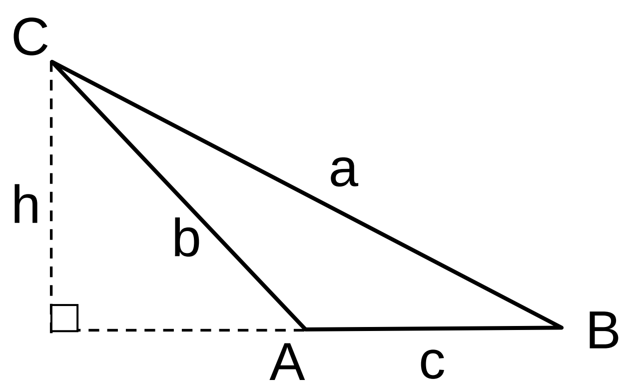
Law of Sines

If *ABC* is a triangle with sides *a*, *b*, and *c*, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{and} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

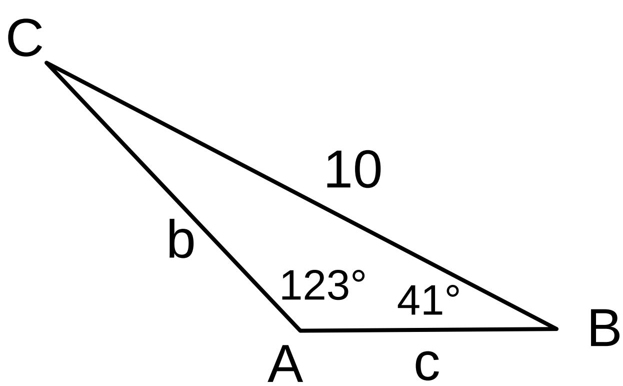


∆ A is acute



∆ A is obtuse

Example: Given A=123°, B=41° and a=10 inches, find c.



$\angle C = 180° - 123° - 41° = 16°$

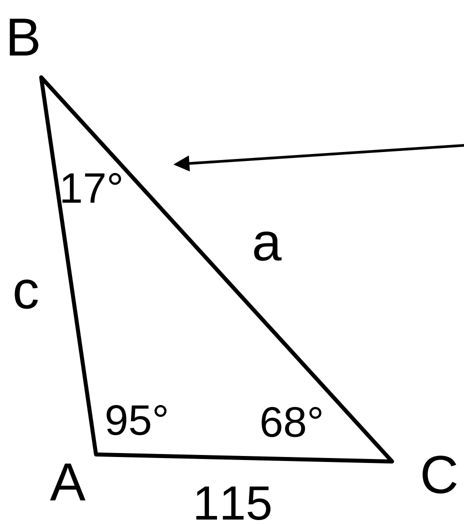
$$\frac{c}{\sin 16°} = \frac{10}{\sin 123°}$$

$$c = \frac{10 \sin 16°}{\sin 123°}$$

$$c \approx 3.29$$

**To use the Law of Sines you must always have one complete pair of angle and opposite side. Then you can solve for any of the other angles or sides.

Example: A triangular plot of land has interior angles *A* = 95° and *C* = 68°. If the side between these angles is 115 yards long, what are the lengths of the other two sides?



$\angle B = 180° - 95° - 68° = 17°$

$$\frac{c}{\sin 68°} = \frac{115}{\sin 17°}$$
$$c = \frac{115 \sin 68°}{\sin 17°}$$
$$c \approx 364.69$$

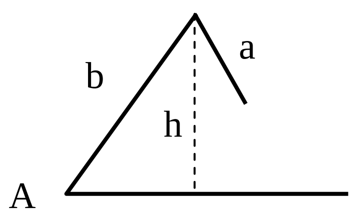
$$\frac{a}{\sin 95°} = \frac{115}{\sin 17°}$$
$$a = \frac{115 \sin 95°}{\sin 17°}$$
$$c \approx 391.84$$

The Ambiguous Case (SSA)

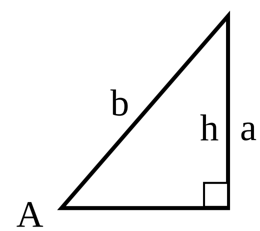
If two sides and one opposite angle are given, three possible situations can occur:

1. no such triangle exists
2. one such triangle exists
3. two distinct triangles may exist

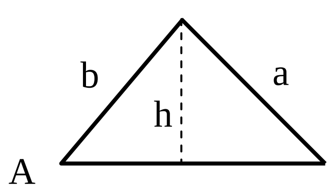
Consider a triangle with a, b, and A are given. ($h = b\sin A$)



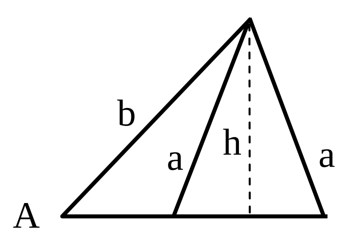
If A is acute, and a<h, then there is no triangle.



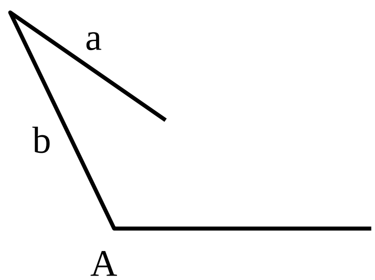
If A is acute, and a=h, then it is a right triangle and there is one triangle.



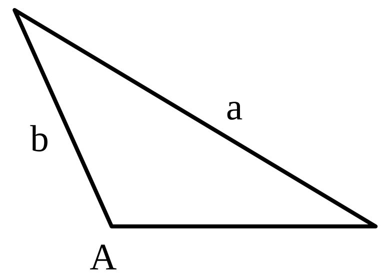
If A is acute, and a>b, then there is one triangle.
(a > h also)



If A is acute, and h<a<b, then there are two triangles.



If A is obtuse, and a#b, then there is <u>no</u> triangle.

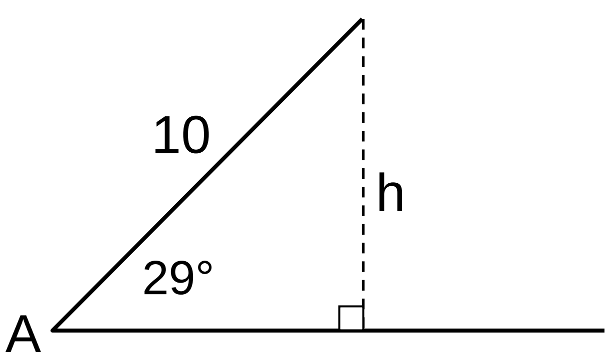


If A is obtuse, and a>b, then there is <u>one</u> triangle.

**If you have an SSA triangle, the angle is acute and the side opposite the known angle is less than the length of the adjacent side, the height of the triangle should be calculated. If the height is less than the length of the opposite side, then two triangles are possible. If the height equals the length of the opposite side, then the triangle is a right triangle.

Example: Given m$\angle A$ = 29°, a = 6, and b = 10, find m$\angle B$.

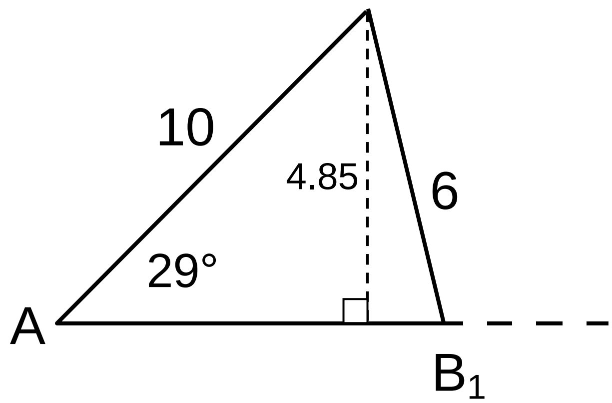
Since $\angle$A is acute, and the side opposite is less than the adjacent side, we need to find the height. Draw the picture and find the height.

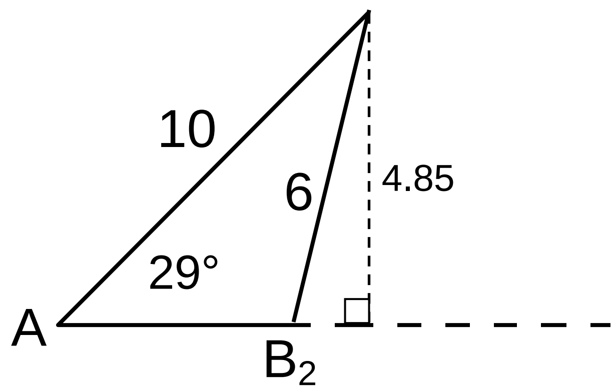


$$\sin 29° = \frac{h}{10}$$

$$h = 10 \sin 29° \approx 4.85$$

Since 6<h<10,
we have 2 triangles.





$$\frac{\sin B}{10} = \frac{\sin 29°}{6}$$

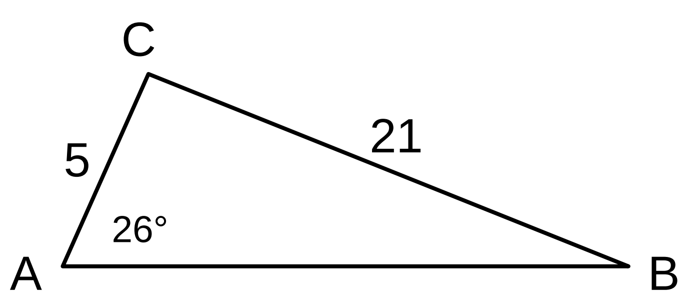
$$\sin B = \frac{10 \sin 29°}{6}$$

$$B \approx 53.9°$$

Since ∠B is acute, it must be ∠B_1. Thinking about the unit circle, and allsintancos, there is another angle with the same sine. It is 180° - 53.9° = 126.1°. Thus, $B_2 \approx$ 126.1°.

Example: Given m∠*A* = 26°, *b* = 5 feet, and *a* = 21 feet, find the other side and the two other angles of the triangle.

Because the side across from the angle is longer than the adjacent side, we know there is one triangle.



$$\frac{\sin B}{5} = \frac{\sin 26°}{21}$$

$$\sin B = \frac{5 \sin 26°}{21}$$

$$B = \sin^{-1}\left(\frac{5 \sin 26°}{21}\right) \approx 6°$$

Since we know m∠A and m∠B, we can find m∠C.

$$m\angle C = 180° - 26° - 6° = 148°$$

Use Law of Sines to find the length of side c.

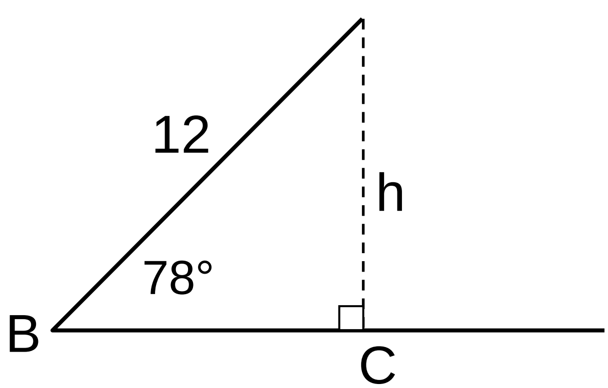
$$\frac{c}{\sin 148°} = \frac{21}{\sin 26°}$$

$$c = \frac{21 \sin 148°}{\sin 26°} \approx 25.4$$

Therefore, $c \approx 25.4$ feet.

Example: Given m∠B = 78°, c = 12, and b = 5, find m∠C.

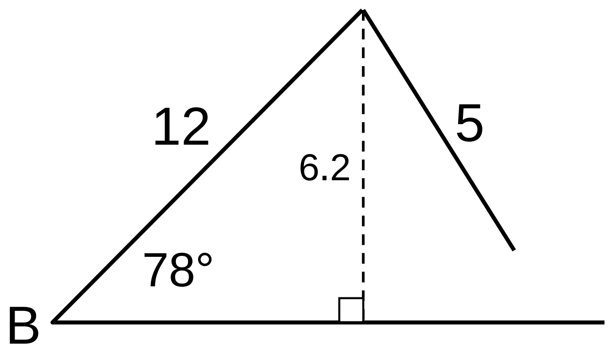
Since the side opposite the known side is less than the adjacent side, we should find the height to see how many triangles we have.



$$\sin 78° = \frac{h}{12}$$

$$h = 12 \sin 78° \approx 11.7$$

Since the height is greater than the opposite side, there is no triangle.



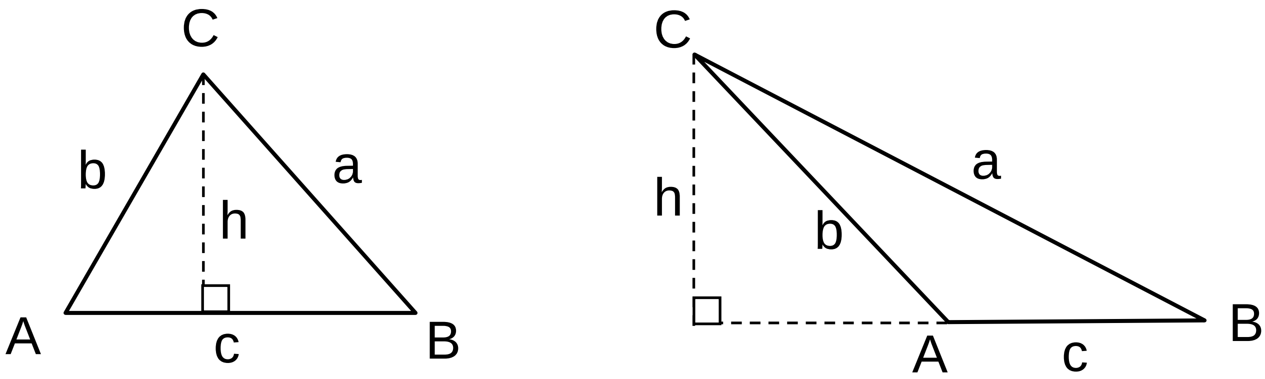
What happens if we try and do the problem without finding the height first?

$$\frac{\sin 78°}{5} = \frac{\sin C}{12}$$
$$\sin C = \frac{12 \sin 78°}{5} \approx 2.35$$

Since the sine of an angle can never be greater than 1.0, this is impossible, and there is no such triangle with these measurements. (If you try to use [SIN] [x^{-1}] on your calculator, you will get an error message.)

Area of an Oblique Triangle

Because we can find the height of triangles using trig, we can use this information to derive a new formula for finding the area of triangles.



We can find the height of a triangle using

$$\sin A = \frac{h}{b}$$
$$h = b\sin A$$

Taking the area formula Area= $\frac{1}{2}$ (base)(height) we get:

$$A = \frac{1}{2}(c)(b\sin A) = \frac{1}{2}bc\sin A$$

By similar methods, we get the following formulas:

Area of an Oblique Triangle

The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle. That is,

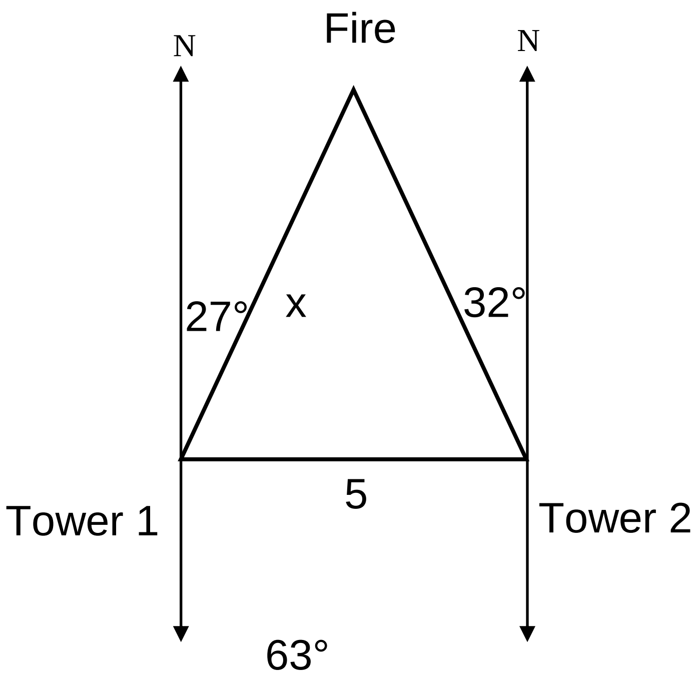
$$\text{Area} = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$$

Example: Find the area of the triangle for which $\angle C = 143°$, $a = 7$ meters, and $b = 18$ meters.

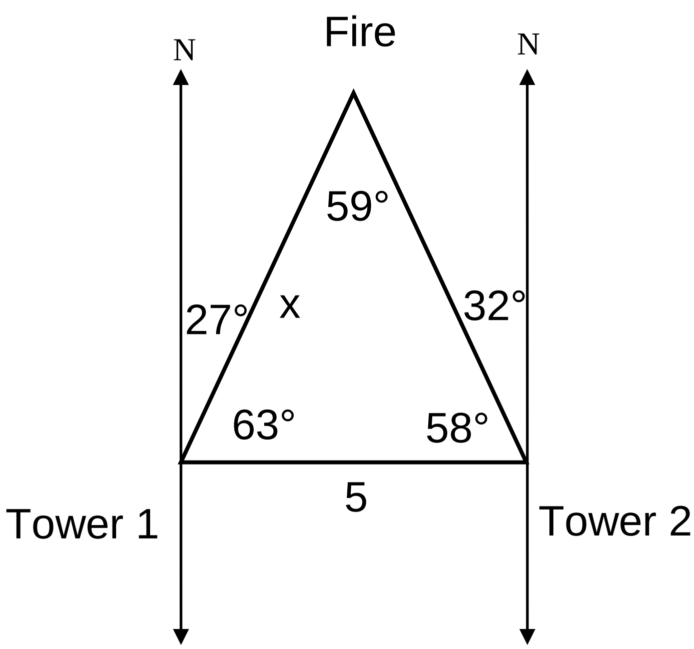
$$\text{Area} = \left(\frac{1}{2}\right)(7)(18)\sin 143° \approx 37.91 \text{ sq. meters}$$

Application

Example: Two fire ranger towers lie on the east-west line and are 5 miles apart. There is a fire with a bearing of N 27°E from tower 1 and N 32° W from tower 2. How far is the fire from tower 1?



Since the tower is on an east-west line, it is perpendicular to the north-south line. So we can find the angle measurements of the lower 2 angles of the triangles, and thus the 3rd angle, also.



Use the Law of Sines to find x.

$$\frac{5}{\sin 59°} = \frac{x}{\sin 58°}$$

$$x = \frac{5\sin 58°}{\sin 59°}$$

$$x \approx 4.9 \text{ miles}$$