Solving Systems of Equations

When we have 2 or more equations and 2 or more unknowns, we use a system of equations to find the solution.

Definition: A solution of a system of equations is an ordered pair that satisfies all of the equations in the system.

Example of a system: $\begin{cases} 2x + y = 5 \\ 3x - 2y = 4 \end{cases}$

The ordered pair (2, 1) is a solution because:

$$\begin{aligned} 2x + y &= 5 \\ 2(2) + 1 &= 5 \\ 4 + 1 &= 5 \\ 5 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x - 2y &= 4 \\ 3(2) - 2(1) &= 4 \\ 6 - 2 &= 4 \\ 4 &= 4 \end{aligned}$$

The ordered pair (2, 1) is a solution to the system because it is a solution for both equations.

The Substitution Method for Solving Systems

Solve the system of equations.

$$\begin{cases} x + y = 5 \\ x = y + 3 \end{cases}$$

Since x = y + 3, we can substitute y + 3 in for x in the top equation without affecting its solution. We get one equation with a single variable. Solve the equation.

$$\begin{aligned} x + y &= 5 \\ (y+3) + y &= 5 \\ 2y + 3 &= 5 \\ 2y &= 2 \\ y &= 1 \end{aligned}$$

Once y is known, you can back-substitute y = 1 into the equation x = y + 3 to find the value of x.

$$\begin{aligned} x &= y + 3 \\ x &= 1 + 3 \\ x &= 4 \end{aligned}$$

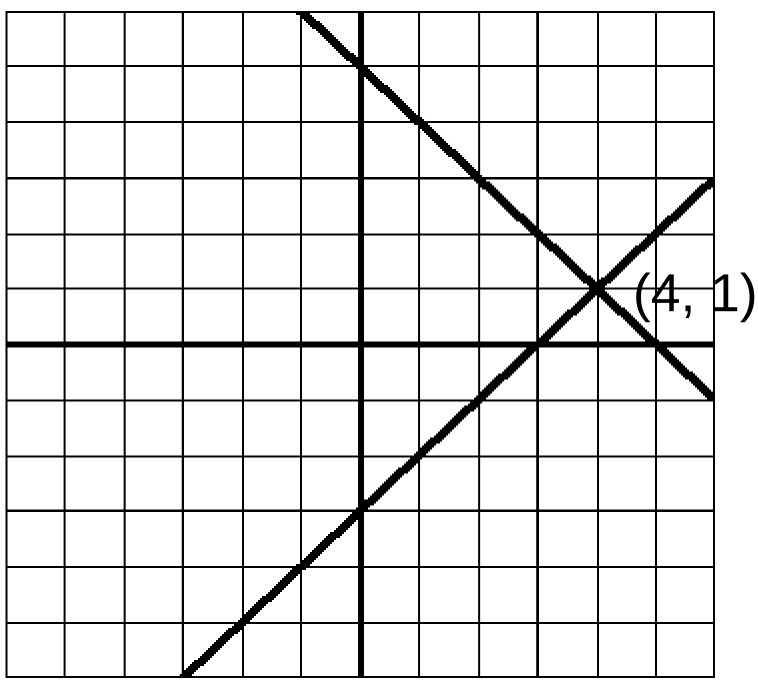
The solution is thus (4, 1). Check this in both of the *original* equations.

<u>Check</u>:

$$\begin{aligned} x + y &= 5 \\ 4 + 1 &= 5 \\ 5 &= 5 \end{aligned} \qquad\qquad \begin{aligned} x &= y + 3 \\ 4 &= 1 + 3 \\ 4 &= 4 \end{aligned}$$

The solution checks in both equations.

Look at the graphs of the lines in the system $\begin{cases} x + y = 5 \\ x = y + 3 \end{cases}$



****The solution to the system is the point where the graphs of the 2 equations intersect.**

Example: Solve the system of equations.

$$\begin{cases} 2x + y = 2 \\ x - 2y = -9 \end{cases}$$

We must begin by solving one of the equations for one of the variables. Solve the first equation for y. (Always solve for the variable that has a coefficient of 1 if possible.)

$$\begin{cases} 2x + y = 2 \rightarrow y = 2 - 2x \\ x - 2y = -9 \end{cases}$$

Substitute and solve the resulting equation.

$$\begin{aligned} x - 2y &= -9 \\ x - 2(2 - 2x) &= -9 \\ x - 4 + 4x &= -9 \end{aligned}$$

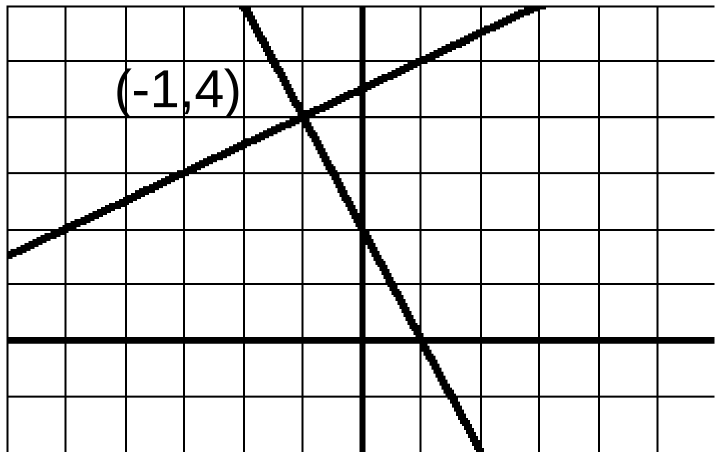
$$\begin{aligned} 5x - 4 &= -9 \\ 5x &= -5 \\ x &= -1 \end{aligned}$$

Back-substitute to find y.

$$\begin{aligned} y &= 2 - 2x \\ y &= 2 - 2(-1) \\ y &= 4 \end{aligned}$$

The solution is (-1, 4).

The point of intersection of the 2 lines is (-1, 4).



<u>Steps for the Substitution Method for Solving Systems:</u>

1. *Solve* one of the equations for one variable in terms of the other.

2. *Substitute* the expression found in Step 1 into the other equation to obtain an equation in one variable.

3. *Solve* the equation obtained in Step 2.

4. *Back-substitute* the value obtained in Step 3 into the expression obtained in Step 1 to find the value of the other variable.

5. *Check* that the solution satisfies each of the original equations.

Example: Solve the system of equations.

$$\begin{cases} x^2 + y^2 = 5 \\ x + y = 1 \end{cases}$$

Solution:

$$\begin{cases} x^2 + y^2 = 5 \\ x + y = 1 \rightarrow x = 1 - y \end{cases}$$

$$x^2 + y^2 = 5$$

$$(1 - y)^2 + y^2 = 5$$

$$1 - 2y + y^2 + y^2 = 5$$

$$2y^2 - 2y + 1 = 5$$

$$2y^2 - 2y - 4 = 0$$

$$2(y^2 - y - 2) = 0$$

$$2(y - 2)(y + 1) = 0$$

$y = 2$ or $y = -1$

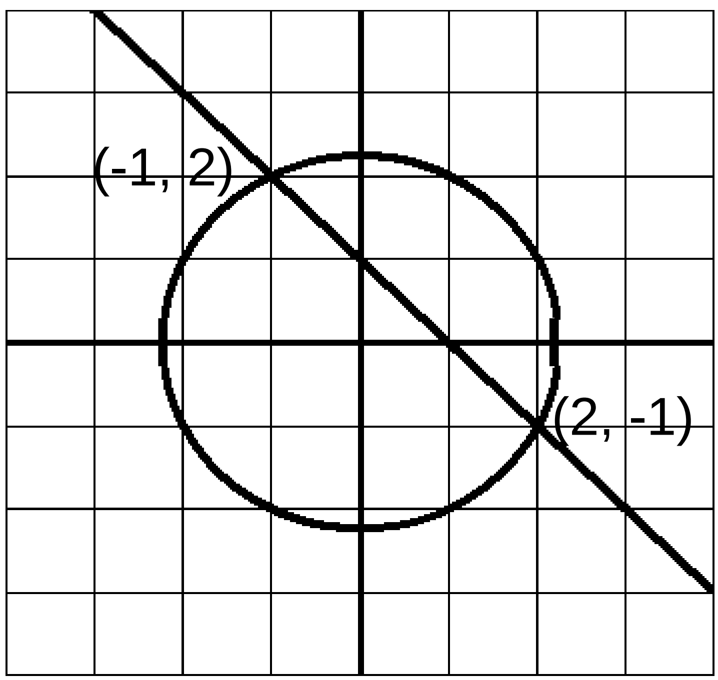
We must back-substitute each of these values of y to solve for the corresponding values of x.

Back-substitute: y = 2 y = -1

$$x = 1 - y \qquad\qquad x = 1 - y$$
$$x = 1 - 2 \qquad\qquad x = 1 - (-1)$$
$$x = -1 \qquad\qquad x = 2$$

The solutions are (-1, 2) and (2, -1).

Look at the graphs of these 2 equations.



The solution to the system is the set of points where the graphs of the 2 equations intersect.

Example: Solve the system of equations.

$$\begin{cases} x^2 + y = 3 \\ y = x + 4 \end{cases}$$

Solution:

$$\begin{cases} x^2 + y = 3 \\ y = x + 4 \end{cases}$$

$$x^2 + y = 3$$

$$x^2 + (x + 4) = 3$$

$$x^2 + x + 1 = 0$$

This does not factor, so use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

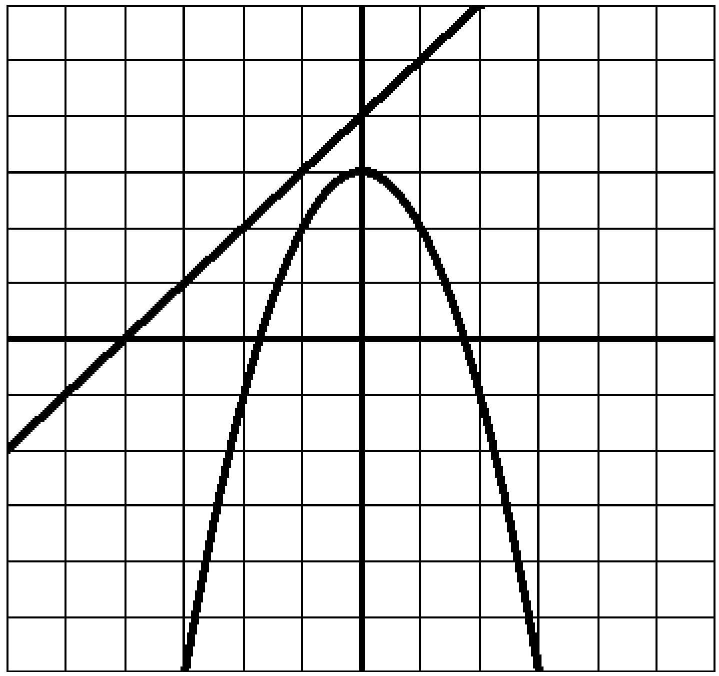
$$x = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$$x = \frac{-1 \pm \sqrt{-3}}{2}$$

This system has no real solutions.

Look at the graph of the system $\begin{cases} x^2 + y = 3 \\ y = x + 4 \end{cases}$



These 2 graphs do not intersect. There is no real solution to this system, as we found by solving algebraically.

<u>A system of 2 equations in 2 unknowns can have:</u>

1. Exactly one real solution
2. More than one real solution
3. No real solution.

Solving Systems Graphically

Example: Solve the system using a graphing calculator.

$$\begin{cases} y = x^2 - x - 1 \\ y = x - 1 \end{cases}$$

Solution:

1. Enter each equation in the [y=] screen.
2. Press [GRAPH].
3. Press [2nd] [CALC] and choose [intersect].
4. The cursor will appear on the graph of the first equation entered. Use the left and right arrows to move the cursor as close the visual point of intersection as possible. Press [ENTER].
5. The cursor will jump to the graph of the 2nd equation. Use the left and right arrows again to move the cursor to the visual point of intersection. Press [ENTER].
6. The calculator screen reads “Guess?” Press [ENTER].
7. The point of intersection is given.

Repeat the process for any other visual points of intersection.

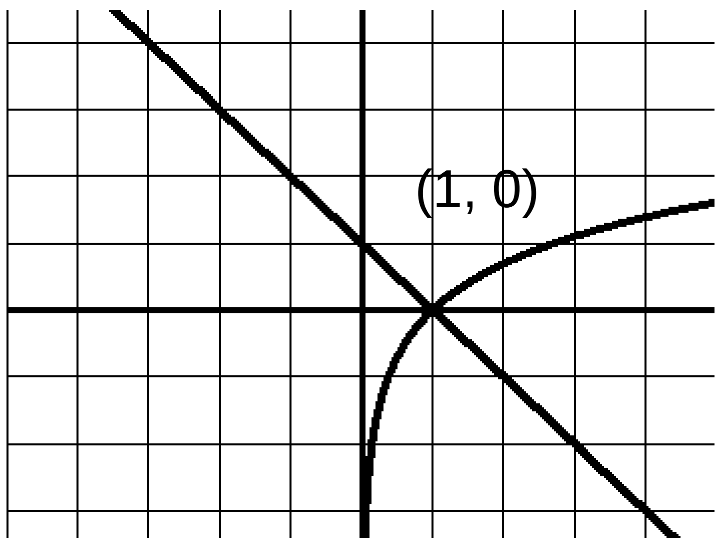
The solution to the system is (0, -1) and (2, 1).

Note: Using the up and down arrows will move the cursor from one equation to the other. The right and left arrows move the cursor along one of the equations.

Example: Solve the system of equations graphically.

$$\begin{cases} y = \ln x \\ x + y = 1 \end{cases}$$

Note: If we tried to solve this by substitution, we would get the equation $x + \ln x = 1$, which is difficult to solve using standard algebraic techniques.



Check:

$$\begin{array}{ll} y = \ln x & x + y = 1 \\ 0 = \ln 1 & 1 + 0 = 1 \\ 0 = 0 & 1 = 1 \end{array}$$

The solution (1, 0) checks in both equations.

Application

Example: A company has fixed monthly manufacturing costs of \$12,000, and it costs \$0.95 to produce each unit of product. The company then sells each unit for \$1.25. How many units must be sold before this company breaks even? (The break-even point is where the cost equals the revenue.)

Solution:

The total cost of producing x units is:

$$\boxed{\text{Total Cost}} = \boxed{\text{Cost per unit}} \cdot \boxed{\text{Number of units}} + \boxed{\text{Fixed cost}}$$

For our problem, $C = 0.95x + 12{,}000$

The total revenue obtained by selling x units is:

$$\boxed{\text{Total Revenue}} = \boxed{\text{Price per unit}} \cdot \boxed{\text{Number of units}}$$

For our problem, $R = 1.25x$

Our system of equations looks like:

$$\begin{cases} C = 0.95x + 12000 \\ R = 1.25x \end{cases}$$

Because the break-even point occurs when R = C, we are really looking at the system:

$$\begin{cases} C = 0.95x + 12000 \\ C = 1.25x \end{cases}$$

Solve this system by substitution.

$$1.25x = 0.95x + 12,000$$
$$0.3x = 12000$$
$$x = 40,000$$

The break-even point occurs when 40,000 units are produced.

Question: How many units must be sold to make a profit of $27,000?

Profit is found by

$$\boxed{\text{Profit}} = \boxed{\text{Revenue}} - \boxed{\text{Cost}}$$

Revenue = the money received from selling the product
Cost = the money spent to produce the product

For our problem $\begin{cases} C = 0.95x + 12000 \\ R = 1.25x \end{cases}$

$$P = R - C$$
$$P = 1.25x - (0.95x + 12{,}000)$$
$$P = 0.3x - 12{,}000$$

We want the profit to be $27,000 so substitute that for P.

$$P = 0.3x - 12{,}000$$
$$27{,}000 = 0.3x - 12{,}000$$
$$39{,}000 = 0.3x$$
$$x = 130{,}000$$

130,000 units must be produced for a profit of $27,000.

Additional Examples:

Example: Solve the system.

$$\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$$

Solution: (-3, 2)

Example: Solve the system.

$$\begin{cases} y = x^3 - 3x^2 + 4 \\ 2x + y = 4 \end{cases}$$

Solution: (0, 4), (1, 2), (2, 0)

Example: Solve the system.

$$\begin{cases} -x + 2y = 1 \\ x - y = 2 \end{cases}$$

Solution: (5, 3)