Two-Variable Linear Systems

Besides solving systems graphically and by substitution, a system can also be solved by the method of elimination (also called the addition method).

The Method of Elimination

Solve the system $\begin{cases} 2x - 3y = 7 \\ 5x + 3y = 0 \end{cases}$

By adding the two equations, we can eliminate the y-terms and obtain a single equation in x. We can then solve for x.

$$\begin{cases} 2x - 3y = 7 \\ 5x + 3y = 0 \end{cases}$$
$$\overline{\quad 7x = 7 \quad}$$
$$\boxed{x = 1}$$

Now back-substitute into either equation to find y.

$$2x - 3y = 7$$
$$2(1) - 3y = 7$$
$$2 - 3y = 7$$
$$-3y = 5$$
$$\boxed{y = \frac{-5}{3}}$$

The solution is $\left(1, \frac{-5}{3}\right)$.

Example: Solve the system of equations.

$$\begin{cases} 3x + 4y = 11 \\ x + 2y = 5 \end{cases}$$

Before we add the equations, we must multiply the 2nd equation by -3.

$$\begin{matrix} \\ (-3) \end{matrix}\begin{cases} 3x + 4y = 11 \\ x + 2y = 5 \end{cases} \Rightarrow \begin{cases} 3x + 4y = 11 \\ -3x - 6y = -15 \end{cases}$$

Now we can add the equations.

$$\begin{cases} 3x + 4y = 11 \\ -3x - 6y = -15 \end{cases}$$
$$-2y = -4$$
$$\boxed{y = 2}$$

Back-substitute into one of the original equations.

$$x + 2y = 5$$
$$x + 2(2) = 5$$
$$x + 4 = 5$$
$$\boxed{x = 1}$$

The solution is (1, 2)

Look at the system $\begin{cases} 3x + 4y = 11 \\ x^2 + y^2 = 10 \end{cases}$

The elimination method does <u>not</u> work for most nonlinear systems of equations. It is best to use the substitution method for nonlinear systems of equations.

Example: Solve the system of equations.

$$\begin{cases} 2x - 3y = -15 \\ 5x + 2y = 10 \end{cases}$$

Before we add the equations, we must multiply the 1st equation by 2 and the 2nd equation by 3 so we can eliminate the y-terms. (Or, we could multiply the 1st equation by -5 and the 2nd equation by 2 if we wanted to eliminate the x-terms.)

$$\begin{matrix} (2) \\ (3) \end{matrix} \begin{cases} 2x - 3y = -15 \\ 5x + 2y = 10 \end{cases} \Rightarrow \begin{cases} 4x - 6y = -30 \\ 15x + 6y = 30 \end{cases}$$

Now we can add the equations.

$$\begin{cases} 4x - 6y = -30 \\ 15x + 6y = 30 \end{cases}$$
$$\overline{\qquad 19x = 0 \qquad}$$
$$\boxed{x = 0}$$

Back-substitute into one of the original equations.

$$\begin{aligned} 2x - 3y &= -15 \\ 2(0) - 3y &= -15 \\ -3y &= -15 \\ \boxed{y = 5} \end{aligned}$$

The solution to the system is (0, 5).

Look again at

$$\begin{matrix} (2) \\ (3) \end{matrix} \begin{cases} 2x - 3y = -15 \\ 5x + 2y = 10 \end{cases} \Rightarrow \begin{cases} 4x - 6y = -30 \\ 15x + 6y = 30 \end{cases}$$

These 2 systems are said to be <u>equivalent</u> because they have exactly the same solution set.

<u>The operations that can be performed on a system of linear equations to produce an equivalent system are</u>:

1. interchanging any two equations
2. multiplying an equation by a nonzero constant
3. adding a multiple of one equation to any other equation in the system

The systems below are all equivalent:

$$\begin{matrix}(2)\\(3)\end{matrix}\begin{cases}2x-3y=-15\\5x+2y=10\end{cases}\Rightarrow\begin{cases}4x-6y=-30\\15x+6y=30\end{cases}\Rightarrow\begin{cases}4x-6y=-30\\19x=0\end{cases}$$

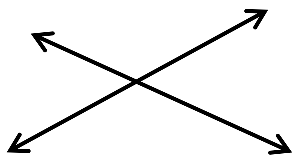
$$\Rightarrow\begin{cases}4x-6y=-30\\x=0\end{cases}\Rightarrow\begin{cases}2x-3y=-15\\x=0\end{cases}\Rightarrow\begin{cases}y=5\\x=0\end{cases}$$

<u>Steps for Solving a System by the Method of Elimination</u>

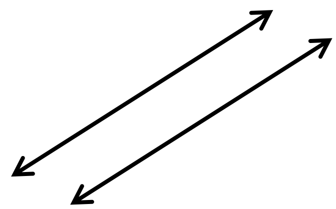
1. *Obtain coefficients* for x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
2. *Add* the equations to eliminate one variable and solve the resulting equation.
3. *Back-substitute* the value obtained in Step 2 into either of the original equations and solve for the other variable.
4. *Check* your solution in both of the original equations.

Graphical Interpretation of Solutions

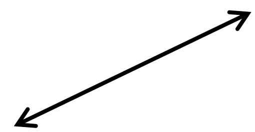
For a system of linear equations in two variables, the number of solutions is one of the following:



- There is exactly one solution.
- The lines intersect at one point.
- The slopes of the lines are different.



- There is no solution.
- The lines are parallel.
- The slopes of the lines are equal.



- There are infinitely many solutions.
- The lines are identical (coincide).
- The slopes of the lines are equal.

Definition: A system of linear equations is consistent if it has at least one solution. It is inconsistent if it has no solution.

Example: Solve the system of equations.

$$\begin{cases} x+3y=5 \\ -2x-6y=1 \end{cases}$$

Multiply the 1st equation by 2.

$$(2)\begin{cases} x+3y=5 \\ -2x-6y=1 \end{cases} \Rightarrow \begin{cases} 2x+6y=10 \\ -2x-6y=1 \end{cases}$$

Now we can add the equations.

$$\begin{array}{r} \begin{cases} 2x+6y=10 \\ -2x-6y=1 \end{cases} \\ \hline 0=11 \end{array}$$

There are no values for x and y that will ever make this true, so there is no solution.

If you get something that is <u>never true</u>, it means that there is <u>no solution</u> and the lines are <u>parallel</u>.

Example: Solve the system of equations.

$$\begin{cases} 0.25x - 0.5y = 1 \\ -x + 2y = -4 \end{cases}$$

First multiply the 1st equation by 100 to clear the decimals.

$$(100)\begin{cases} 0.25x - 0.5y = 1 \\ -x + 2y = -4 \end{cases} \Rightarrow \begin{cases} 25x - 50y = 100 \\ -x + 2y = -4 \end{cases}$$

Now multiply the bottom equation by 25.

$$\begin{matrix} \\ (25) \end{matrix}\begin{cases} 25x - 50y = 100 \\ -x + 2y = -4 \end{cases} \Rightarrow \begin{cases} 25x - 50y = 100 \\ -25x + 50y = -100 \end{cases}$$

Now we can add the equations.

$$\begin{cases} 25x - 50y = 100 \\ -25x + 50y = -100 \end{cases}$$
$$0 = 0$$

These 2 equations are equivalent, so there are infinitely many solutions.

If you get something that is <u>always true</u>, it means that there are <u>infinitely many solutions</u> and the lines are <u>the same line.</u>

Applications

Example: A man in a boat can row 8 miles downstream in 1 hour. He can row 6 miles upstream in 3 hours. How fast can the man row in still water, and what is the rate of the current?

Let r = rowing rate in still water
Let c = current rate of water

	rate ·	time	= distance
downstream	r+c	1	8
upstream	r-c	3	6

Our 2 equations can be put in the system:

$$\begin{cases}(r+c)(1)=8\\(r-c)(3)=6\end{cases} \Rightarrow \begin{cases}r+c=8\\3r-3c=6\end{cases}$$

Solve the system.

$$(3)\begin{cases}r+c=8\\3r-3c=6\end{cases} \Rightarrow \begin{cases}3r+3c=24\\3r-3c=6\end{cases}$$

$$6r=30 \qquad r+c=8$$

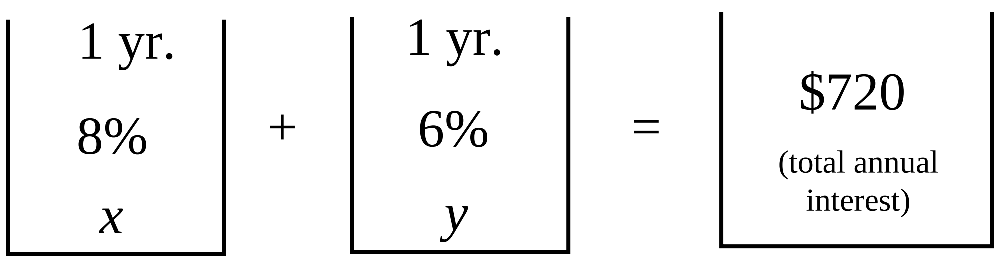
$$\boxed{r=5} \qquad \boxed{c=3}$$

The current is 3 mph and the rowing rate is 5 mph.

Example: You have \$10,000 to invest in two simple interest funds. One pays 8% and the other 6%. How much should you invest in each account so that the total annual interest is \$720?

This problem is easiest done with interest buckets. Each bucket represents interest. Remember that

interest = principal · rate · time (in years)



We need 2 equations.

$$\begin{cases} x + y = 10{,}000 \\ 0.08x + 0.06y = 720 \end{cases}$$

$x + y = 10{,}000$ ← Amount invested

$0.08x + 0.06y = 720$ ← Interest (Multiply the buckets)

$$(100)\begin{cases} x + y = 10{,}000 \\ 0.08x + 0.06y = 720 \end{cases} \Rightarrow \begin{cases} x + y = 10{,}000 \\ 8x + 6y = 72{,}000 \end{cases}$$

$$(-6)\begin{cases} x + y = 10{,}000 \\ 8x + 6y = 72{,}000 \end{cases} \Rightarrow \begin{cases} -6x - 6y = -60{,}000 \\ 8x + 6y = 72{,}000 \end{cases}$$

$$2x = 12{,}000$$

$$\boxed{x = 6{,}000}$$

$$x + y = 10{,}000$$

$$6{,}000 + y = 10{,}000$$

$$\boxed{y = 4000}$$

$6,000 was invested at 8% and $4000 was invested at 6%.

Additional Examples:

Example: Solve the system.

$$\begin{cases} -x+3y=17 \\ 4x+3y=7 \end{cases}$$

Solution: (-2, 5)

Example: Solve the system.

$$\begin{cases} 2x+5y=8 \\ 5x+8y=10 \end{cases}$$

Solution: $\left(\frac{-14}{9},\frac{20}{9}\right)$

Example: Solve the system.

$$\begin{cases} \frac{x-1}{2}+\frac{y+2}{3}=4 \\ x-2y=5 \end{cases}$$

Solution: (7, 1)