Multivariable Linear Systems

Look at the following equivalent systems:

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y \qquad = -4 \\ 2x - 5y + 5z = 17 \end{cases} \qquad \begin{cases} x - 2y + 3z = 9 \\ \qquad y + 3z = 5 \\ \qquad\qquad z = 2 \end{cases}$$

The 2nd system is in <u>row-echelon form</u>. This means that it has a "stair-step" pattern with leading coefficients of 1. A system in row-echelon form is easier to solve. Use back-substitution.

$$\begin{aligned} y + 3z &= 5 \\ y + 3(2) &= 5 \\ y + 6 &= 5 \\ \boxed{y = -1} \end{aligned} \qquad \begin{aligned} x - 2y + 3z &= 9 \\ x - 2(-1) + 3(2) &= 9 \\ x + 2 + 6 &= 9 \\ \boxed{x = 1} \end{aligned}$$

The solution to the system is (1, -1, 2). This is called an <u>ordered triple</u>.

Solving a system of equations by transforming it into row-echelon form is called <u>Gaussian Elimination</u>**.**

Operations That Produce Equivalent Systems

Each of the following **row operations** will transform a system of equations into an *equivalent* system of equations.

1. Interchange two equations.
2. Multiply any of the equations by a nonzero constant.
3. Add a multiple of one equation in the system to another equation to replace the latter equation.

Example: Solve the system using Gaussian Elimination

$$(-2)\begin{cases} x-2y=1 \\ 2x-3y=6 \end{cases} \quad \Rightarrow \quad \begin{cases} x-2y=1 \\ \boxed{y=4} \end{cases}$$

$$-2x+4y=-2$$

Now use back-substitution.

$$\begin{aligned} x-2y&=1 \\ x-2(4)&=1 \\ x-8&=1 \\ \boxed{x=9}& \end{aligned}$$

The solution is (9, 4).

Example: Solve the system.

$$\begin{cases} 2x+4y+z=1 \\ x-2y-3z=2 \\ x+y-z=-1 \end{cases} \Rightarrow$$

$$(-1)\begin{cases} 2x+4y+z=1 \\ x-2y-3z=2 \\ x+y-z=-1 \end{cases}$$

$$\rightarrow -x+2y+3z=-2$$

$$\begin{cases} 2x+4y+z=1 \\ x-2y-3z=2 \\ \quad 3y+2z=-3 \end{cases} \leftarrow$$

Add -1 times the 2nd equation to the 3rd equation, and replace the 3rd equation with the sum equation.

$$(-2)\begin{cases} 2x+4y+z=1 \\ x-2y-3z=2 \\ \quad 3y+2z=-3 \end{cases}$$

$$\rightarrow -2x+4y+6z=-4$$

$$\begin{cases} x-2y-3z=2 \\ \quad 8y+7z=-3 \\ \quad 3y+2z=-3 \end{cases} \leftarrow$$

Add -2 times the 2nd equation to the 1st equation, and replace the 1st equation with the sum equation. Rearrange equations.

$$\begin{array}{r} \\ (-3) \\ (8) \end{array}\begin{cases} x - 2y - 3z = 2 \\ \quad 8y + 7z = -3 \\ \quad 3y + 2z = -3 \end{cases}$$

$$-24y - 21z = -9$$
$$24y + 16z = -6$$

$$\begin{cases} x - 2y - 3z = 2 \\ \quad 3y + 2z = -3 \\ \quad\quad -5z = -15 \end{cases}$$

Add -3 times the 2nd equation to 8 times the 3rd equation, and replace the 2nd equation with the sum equation. Rearrange equations.

Solve for z and then back-substitute.

$$-5z = -15$$
$$z = 3$$

$$3y + 2z = -3$$
$$3y + 2(3) = -3$$
$$3y + 6 = -3$$
$$3y = -9$$
$$y = -3$$

$$x - 2y - 3z = 2$$
$$x - 2(-3) - 3(3) = 2$$
$$x + 6 - 9 = 2$$
$$x - 3 = 2$$
$$x = 5$$

The solution is (5, -3, 3).

Example: Solve the system.

$$\begin{cases} x-3y+2z=1 \\ 2x-5y+z=-5 \\ 3x+y-2z=-1 \end{cases} \Rightarrow \begin{cases} x-3y+2z=1 \\ y-3z=-7 \\ 10y-8z=-4 \end{cases}$$

We added -2 times the first equation to the 2nd equation.
We added -3 times the 1st equation to the 3rd equation.

$$\begin{cases} x-3y+2z=1 \\ y-3z=-7 \\ 22z=66 \end{cases} \Rightarrow \begin{cases} x-3y+2z=1 \\ y-3z=-7 \\ z=3 \end{cases}$$

We added -10 times the 2nd equation to the 3rd equation.
We divided through the 3rd equation by 22.

By back-substitution we get the solution (1, 2, 3).

Example: Solve the system.

$$\begin{cases} x+3y-2z=1 \\ y+4z=-1 \\ 2y+8z=3 \end{cases} \Rightarrow \begin{cases} x+3y-2z=1 \\ y+4z=-1 \\ 0=5 \end{cases}$$

If we get something like 0=5, which is never true, there is no solution.

Example: Solve the system.

$$\begin{cases} x+y+z=2 \\ y+2z=1 \\ -2y-4z=-2 \end{cases} \Rightarrow \begin{cases} x+y+z=2 \\ y+2z=1 \\ 0=0 \end{cases}$$

Since 0=0 is always true, there are infinitely many solutions. What we can do when there are infinitely many solutions (for any system) is let z = a (an arbitrary variable). Then back-substitute.

Let z = a

$$y+2z=1$$
$$y+2a=1$$
$$y=1-2a$$

$$x+y+z=2$$
$$x+(1-2a)+a=2$$
$$x-a+1=2$$
$$\boxed{x=a+1}$$

The solution is (a+1, 1-2a, a).

Nonsquare Systems

A nonsquare system is one in which the number of unknowns is not the same as the number of equations.

Example: Solve the system.

$$\begin{cases} x-2y+z=2 \\ 2x-y-z=1 \end{cases} \Rightarrow \begin{cases} x-2y+z=2 \\ \quad 3y-3z=-3 \end{cases} \Rightarrow \begin{cases} x-2y+z=2 \\ \quad\quad y-z=-1 \end{cases}$$

Add -2 times the 1st equation to the 2nd equation. Then multiply the 3rd equation by 1/3.

Solve for y.

$$y=z-1$$

Put that in the 1st equation and solve for x.

$$x-2y+z=2$$

$$x-2(z-1)+z=2$$

$$x-2z+2+z=2$$

$$x=z$$

Our solution is $(z, z\text{-}1, z)$. Since z can be any number, use a instead to get the solution $(a, a\text{-}1, a)$.

Example: The following equivalent system is obtained during the course of Gaussian elimination. Write the solution of the system.

$$\begin{cases} x + 2y - z = 4 \\ y + 2z = 8 \\ 0 = 0 \end{cases}$$

Let z = a. Then back-substitute into the 2nd equation.

$$y + 2z = 8 \qquad x + 2y - z = 4$$
$$y + 2a = 8 \qquad x + 2(8 - 2a) - a = 4$$
$$y = 8 - 2a \qquad x = 5a - 12$$

The solution is (5a-12, 8-2a, a).

Example: Solve the following system.

$$\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$$

Add -5 times the first 1st equation to the 2nd equation. Replace the 2nd equation with the sum.

$$\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$$

$-5x+15y$ $-10z$ $= -90$

$$\begin{cases} x - 3y + 2z = 18 \\ \quad 2y + 2z = -10 \end{cases}$$ Divide the 2nd equation by 2.

$$\begin{cases} x - 3y + 2z = 18 \\ \quad y + z = -5 \end{cases}$$

$$y = -5 - z$$

$$x - 3y + 2z = 18$$
$$x - 3(-5 - z) + 2z = 18$$
$$x = 3 - 5z$$

The solution is ($-5z+3$, $-z - 5$, z).

Graphical Interpretation of Solutions

For a system of linear equations in three variables, the number of solutions is one of the following:

- There is exactly one solution (a point).
- There are infinitely many solutions (either a single line, or they all name the same plane).
- There is no solution.

Applications

Problem: Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points (1, 6), (-1, 4), and (2, 13).

We get 3 equations that must be true, since all 3 points must work in the equation.

$$\begin{aligned} 6 &= a(1)^2 + b(1) + c \\ 4 &= a(-1)^2 + b(-1) + c \\ 13 &= a(2)^2 + b(2) + c \end{aligned} \quad \rightarrow \quad \begin{aligned} 6 &= a + b + c \\ 4 &= a - b + c \\ 13 &= 4a + 2b + c \end{aligned}$$

From this we can get the following system:

$$\begin{cases} a+b+c=6 \\ a-b+c=4 \\ 4a+2b+c=13 \end{cases}$$

Solving the system we get (2, 1, 3). Putting them back into our quadratic equation we get:

$$y = 2x^2 + x + 3$$

Problem: During the second game of the 2002 Western Conference finals, the Los Angeles Lakers scored a total of 90 points, resulting from a combination of three-point baskets, two-point baskets, and one-point free-throws. There were 11 times as many two-point baskets as three-point baskets and five times as many free-throws as three-point baskets. What combination of scoring accounted for the Lakers' 90 points?

Let x = number of 3-point baskets
Let y = number of 2-point baskets
Let z = number of 1-point free-throws

$$\begin{cases} 3x + 2y + z = 90 \\ y = 11x \\ z = 5x \end{cases}$$

Substitute for y and z in the 1st equation.

$$\begin{aligned} 3x + 2y + z &= 90 \\ 3x + 2(11x) + 5x &= 90 \\ 3x + 22x + 5x &= 90 \\ 30x &= 90 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} y &= 11x \\ y &= 11(3) \\ y &= 33 \end{aligned}$$

$$\begin{aligned} z &= 5x \\ z &= 5(3) \\ z &= 15 \end{aligned}$$

<u>Solution</u>: <u>3</u> 3-point, <u>33</u> 2-point, <u>15</u> 1-point baskets

Problem: A small corporation borrowed $800,000 to expand its line of toys. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate if the annual interest owed was $67,000 and the amount borrowed at 8% was five times the amount borrowed at 10%?

Let x = amount at 8%
Let y = amount at 9%
Let z = amount at 10%

$$\begin{cases} x + y + z = 800{,}000 \\ 0.08x + 0.09y + 0.10z = 67{,}000 \\ x = 5z \end{cases}$$

Multiply 100 times the 2nd equation.

$$\begin{cases} x + y + z = 800{,}000 \\ 8x + 9y + 10z = 6{,}700{,}000 \\ x \quad - 5z = 0 \end{cases}$$

Multiply -1 times the 3rd equation and add it to the 1st equation.

$$\begin{cases} 8x + 9y + 10z = 6{,}700{,}000 \\ y + 6z = 800{,}000 \\ x \quad - 5z = 0 \end{cases}$$

Multiply -8 times the 3rd equation and add it to the 1st equation.

$$\begin{cases} x \qquad\quad -5z = 0 \\ \quad 9y + 50z = 6{,}700{,}000 \\ \qquad y + 6z = 800{,}000 \end{cases}$$

Multiply -9 times the 2nd equation and add it to the 3rd equation.

$$\begin{cases} x \qquad\quad -5z = 0 \\ \qquad y + 6z = 800{,}000 \\ \qquad\quad -4z = -500{,}000 \end{cases}$$

Solve for z and then back-substitute.

$$z = 125{,}000$$

$$y + 6z = 800{,}000$$
$$y + 6(125{,}000) = 800{,}000$$
$$y + 750{,}000 = 800{,}000$$
$$y = 50{,}000$$

$$x - 5z = 0$$
$$x - 5(125{,}000) = 0$$
$$x = 625{,}000$$

<u>Solution</u>: x = $625,000 at 8%
y = $50,000 at 9%
z = $125,000 at 10%