Operations with Matrices

Equality of Matrices

There are three ways to represent a matrix.

1. A matrix can be denoted by an uppercase letter, such as A, B, or C.
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Definition: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are <u>equal</u> if they have the same order (m x n) and $[a_{ij}] = [b_{ij}]$ for all i = 1, 2, …, m and j = 1, 2, …, n. In other words, if all of the corresponding entries are equal.

$$\begin{bmatrix} 2 & -1 \\ \sqrt{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0.5 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \neq \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 0 & 0 \end{bmatrix}$$

Testing Matrix Equality using a Graphing Calculator

1. Press [2nd] [MATRX] [A]
2. Press [2nd] [TEST] [=]
3. Press [2nd] [MATRX] [B] [ENTER]

- If the matrices are of the same order, and all corresponding elements are equal, then the calculator will return the value “1.”
- If the matrices are of the same order, and all corresponding elements are not all equal, then the calculator will return the value “0.”
- If the matrices do not have the same dimensions, you will get and error message that says “DIM MISMATCH”

Copying a Matrix

To place the contents of matrix A into the matrix B, do the following:

1. Press [2nd] [MATRX] [A]
2. Press [STO▹]
3. Press [2nd] [MATRX] [B] [ENTER]

A and B are now identical matrices.

Matrix Addition

Definition of Matrix Addition:

If A = [a_{ij}] and B = [b_{ij}] are matrices of order m x n, their sum is the m x n matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

*The sum of two matrices of different orders is undefined.

Example: Find the following sums.

a) $\begin{bmatrix} 2 & -5 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 8 & -9 \\ 1 & -3 \end{bmatrix}$

b) $\begin{bmatrix} 4 & -5 \\ -6 & 2 \\ 3 & 8 \end{bmatrix} + \begin{bmatrix} 8 & 9 \\ 6 & -1 \\ 3 & -2 \end{bmatrix}$

Solutions:

$\begin{bmatrix} 10 & -14 \\ 5 & 0 \end{bmatrix}$

$\begin{bmatrix} 12 & 4 \\ 0 & 1 \\ 6 & 6 \end{bmatrix}$

Scalar Multiplication

In operations with matrices, numbers are usually referred to as scalars.

Definition of Scalar Multiplication

If A = $[a_{ij}]$ is an m x n matrix and c is a scalar, the scalar multiple of A by c is the m x n matrix given by

$$cA = [ca_{ij}]$$

Definition: The symbol –A represents the additive inverse of A and equals (-1)A. Moreover, A – B = A + (-B).

Example: Consider the matrices:

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 8 & -6 \\ 5 & -7 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 6 & 2 \\ 4 & -3 & 5 \\ 9 & -2 & 3 \end{bmatrix}$$

Find the following:

a) 2A

$$2A = 2\begin{bmatrix} 4 & -2 & 3 \\ 1 & 8 & -6 \\ 5 & -7 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 6 \\ 2 & 16 & -12 \\ 10 & -14 & 2 \end{bmatrix}$$

b) -B

$$-B = (-1)B = (-1)\begin{bmatrix} -1 & 6 & 2 \\ 4 & -3 & 5 \\ 9 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -6 & -2 \\ -4 & 3 & -5 \\ -9 & 2 & -3 \end{bmatrix}$$

c) A – B

$$A - B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 8 & -6 \\ 5 & -7 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 6 & 2 \\ 4 & -3 & 5 \\ 9 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -8 & 1 \\ -3 & 11 & -11 \\ -4 & -5 & -2 \end{bmatrix}$$

Matrix Operations on the Graphing Calculator

Adding Matrices

If the matrices have the same dimensions, they can be added. To add matrix A and matrix B, do the following:

1. Press [2nd] [MATRX] [A]
2. Press [+]
3. Press [2nd] [MATRX] [B] [ENTER]

The resulting matrix is A + B.

***The same method is used for subtracting matrices.

Note: If the matrices do not have the same dimensions, you will get and error message that says "DIM MISMATCH"

Scalar Multiplication

To multiply a scalar times the matrix A, do the following:

1. Enter the scalar value
2. Press [*] [2nd] [MATRX] [A] [ENTER]

Note: You can multiply the scalar before or after the matrix.

Negating a Matrix

To change the signs on all of the elements of a matrix, do the following:

1. Enter the negation symbol [(-)]
2. Press [2nd] [MATRX] [A] [ENTER]

Example: Enter the following matrices on your calculator:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 7 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix}$$

Find the following:

a) A + B

$$A + B = \begin{bmatrix} 3 & 6 & 9 \\ -1 & 2 & -1 \end{bmatrix}$$

b) 3B

$$3B = \begin{bmatrix} 3 & 6 & 9 \\ -12 & -15 & -18 \end{bmatrix}$$

c) -A

$$-A = \begin{bmatrix} -2 & -4 & -6 \\ -3 & -7 & -5 \end{bmatrix}$$

d) 2A - B

$$2A - B = \begin{bmatrix} 3 & 6 & 9 \\ 10 & 19 & 16 \end{bmatrix}$$

Definition: The zero matrix is the *m* x *n* matrix given by *O* = [0]. The zero matrix can be any size, and consists entirely of zeros. The zero matrix is also the additive identity for the set of all *m* x *n* matrices.

Properties of Matrix Addition and Scalar Multiplication

Let *A*, *B,* and *C* be *m* x *n* matrices and let *c* and *d* be scalars.

1. $A + O = O + A = A$
2. $A + B = B + A$
3. $A + (B + C) = (A + B) + C$
4. $(cd)A = c(dA)$
5. $1A = A$
6. $c(A + B) = cA + cB$
7. $(c + d)A = cA + dA$

Note: #6 and #7 also mean that we can factor out a common factor for any matrix.

The algebra of real numbers and the algebra of matrices have many similarities.

Example: Solve the matrix equation $A - 2X = B$, where

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & -6 \\ 5 & 1 \end{bmatrix}$$

Solving for X we get $X = -\frac{1}{2}(B - A)$.

$$X = -\frac{1}{2}\left(\begin{bmatrix} -3 & -6 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}\right)$$

$$= -\frac{1}{2}\left(\begin{bmatrix} -5 & -5 \\ 1 & -2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{-1}{2} & 1 \end{bmatrix}$$

Example: Solve $B - X = 2A$ using the matrices A and B as given above.

$$X = B - 2A$$

$$= \begin{bmatrix} -3 & -6 \\ 5 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -6 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -4 \\ -3 & -5 \end{bmatrix}$$

Matrix Multiplication

Definition of Matrix Multiplication:

If $A = [a_{ij}]$ is an m x n matrix and $B = [b_{ij}]$ is an n x p matrix, the product AB is an m x p matrix given by

$$AB = [c_{ij}]$$

where $c_{ij} = a_{i1}b_{i1} + a_{i2}b_{i2} + a_{i3}b_{i3} + \ldots + a_{in}b_{nj}.$

What we are doing is taking the elements in the ith row of A, multiplying them by the corresponding elements in the jth column of B, and then summing these products.

Example: Multiply $\begin{bmatrix} -1 & 6 \\ 3 & -2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -3 & 2 \\ -5 & 7 \end{bmatrix}$

For our product, we will multiply each row of the first matrix times each column of the 2nd matrix.

$$= \begin{bmatrix} (-1)(-3)+(6)(-5) & (-1)(2)+(6)(7) \\ (3)(-3)+(-2)(-5) & (3)(2)+(-2)(7) \\ (1)(-3)+(4)(-5) & (1)(2)+(4)(7) \end{bmatrix} = \begin{bmatrix} -27 & 40 \\ 1 & -8 \\ -23 & 30 \end{bmatrix}$$

This entry in *row 2, column 1* came from multiplying *row 2* of the first matrix times *column 1* of the 2nd matrix.

Note: In order for the product to be defined, the number of columns of the first matrix must equal the number of rows of the 2nd matrix.

A x B = AB

m x n n x p m x p

Equal

Dimensions of AB

Example: Find the following products.

a) $\begin{bmatrix} -1 & 0 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 2 & -7 \\ 32 & -12 & 48 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 6 & 3 \\ 2 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 5 \\ 4 & 3 & -2 \\ -1 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 11 & -1 \\ 21 & 6 & -6 \\ -9 & 6 & 8 \end{bmatrix}$

c) $\begin{bmatrix} 2 & -5 \\ 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} -9 & -32 \\ 48 & 33 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 3 \\ 6 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 4 & 7 & 7 \end{bmatrix}$ = undefined

e) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 4 & 2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 16 \end{bmatrix}$

f) $$\begin{bmatrix} 2 & -3 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \end{bmatrix}$$

g) $$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & 4 \\ 4 & -6 & 2 & 8 \\ -2 & 3 & -1 & -4 \\ 10 & -15 & 5 & 20 \end{bmatrix}$$

*Notice that for **f)** and **g)** that we did not get the same answer. Even if AB and BA are defined, **matrix multiplication is, in general, not commutative**.

Matrix Multiplication on the Graphing Calculator

If the number of columns of the first matrix equals the number of rows of the 2nd matrix, the matrices can be multiplied. (An error message will result otherwise.)

1. Press [2nd] [MATRX] [A]
2. Press [*]
3. Press [2nd] [MATRX] [B] [ENTER]

The resulting matrix is A · B.

Example: Enter the following matrices on your calculator:

$$A = \begin{bmatrix} -1 & 6 \\ 3 & -2 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 2 \\ -5 & 7 \end{bmatrix}$$

Find AB.

$$AB = \begin{bmatrix} -27 & 40 \\ 1 & -8 \\ -23 & 30 \end{bmatrix}$$

Definition: The <u>identity matrix</u> of order n is an n x n matrix that consists of 1's on its main diagonal and 0's everywhere else. It is denoted I_n. If it is already understood that the matrix is square, we can refer to it as simply I.

examples: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Example: Multiply $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (3)(1)+(4)(0) & (3)(0)+(4)(1) \\ (5)(1)+(6)(0) & (5)(0)+(6)(1) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Multiplying by the identity matrix gives back the matrix we started with.

Properties of Matrix Multiplication

Let A, B, and C be matrices and let *c* be a scalar.

1. $AI_n = I_nA = A$
2. $A(BC) = (AB)C$
3. $A(B + C) = AB + AC$
4. $(A + B)C = AC + BC$
5. $c(AB) = (cA)B = A(cB)$

Applications

Notice how the system $\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=12 \end{cases}$

Can be written as the matrix equation AX = B, where A is the coefficient matrix and X and B are column matrices.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1x+\ 2y+\ 3z \\ 5x+\ 6y+\ 7z \\ 9x+10y+11z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

$$\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=12 \end{cases}$$

Example: Write the following system of equations as a matrix equation AX = B.

$$\begin{cases} 2x + 3y + 4z = 5 \\ 3x + 9y - 5z = 0 \\ 5x - 3y + 5z = 9 \end{cases}$$

<u>Solution</u>: The matrix equation is:

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 9 & -5 \\ 5 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix}$$

Example: Write the following system of equations as a matrix equation AX = B. Then use Gauss-Jordon elimination on the augmented matrix [A:B] to solve for the matrix X.

$$\begin{cases} 2x_1 - x_2 + 3x_3 = -11 \\ x_1 \qquad - 3x_3 = -1 \\ -x_1 + 4x_2 + 2x_3 = 2 \end{cases}$$

<u>Solution</u>:

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -3 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ -1 \\ 2 \end{bmatrix}$$

The augmented matrix [A:B] is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & -11 \\ 1 & 0 & -3 & -1 \\ -1 & 4 & 2 & 2 \end{array}\right] \quad \text{or} \quad \begin{bmatrix} 2 & -1 & 3 & -11 \\ 1 & 0 & -3 & -1 \\ -1 & 4 & 2 & 2 \end{bmatrix}$$

Solving gives:

$$\begin{bmatrix} 2 & -1 & 3 & -11 \\ 1 & 0 & -3 & -1 \\ -1 & 4 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 4 & -1 & 1 \\ 0 & 7 & 7 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 4 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 4 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The solution is X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix}$

$x_1 = -4$
$x_2 = 0$
$x_3 = -1$

Example: Two tennis teams submit equipment requests to their sponsors.

	Women's Team	*Men's Team*
Balls	50	48
Rackets	12	15
Shoes	15	18

Each can of balls costs \$5, each racket costs \$129, and each pair of shoes costs \$79. Use matrix multiplication to find the total cost of equipment for each team.

Solution:

$$\begin{bmatrix} 5 & 129 & 79 \end{bmatrix} \cdot \begin{bmatrix} 50 & 48 \\ 12 & 15 \\ 15 & 18 \end{bmatrix} = \begin{bmatrix} 2983 & 3597 \end{bmatrix}$$

So, the total cost of equipment for the women's team is \$2,983 and the total cost of equipment for the men's team is \$3,597.