Arithmetic Sequences and Partial Sums

A sequence whose consecutive terms have a common difference is called an arithmetic sequence.

Definition of Arithmetic Sequence

A sequence is arithmetic if the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots a_n, \ldots$$

is arithmetic if there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = d$$

and so on. The number d is the common difference of the arithmetic sequence.

Example: Find the common difference in the following arithmetic sequences.

a) 5, 8, 11, 14, …

$$d = a_2 - a_1 = 8 - 5 = 3$$

b) 15, 11, 7, 3, …

$$d = a_2 - a_1 = 11 - 15 = -4$$

Example: Determine which of the following are arithmetic sequences and find the common difference if they are.

a) 3, 5, 7, 9, 11,…

yes, $d = 2$

b) 3, 6, 12, 24, 48, …

no

c) -3, 6, -9, 12, -15, …

no

d) 5, 0, -5, -10, -15, …

yes, $d = -5$

e) 1, 3, 6, 10, 15, 21, …

no

f) $3, \frac{5}{2}, 2, \frac{3}{2}, 1, ...$

yes, $d = -\frac{1}{2}$

Examples of Arithmetic Sequences

The sequence with nth term $a_n = 4n + 3$.

$7, 11, 15, 19, \ldots..4n+3, \ldots \qquad d = 4$

The sequence with nth term $a_n = 5 - 3n$.

$2, -1, -4, -7, -10, \ldots.\ 5 - 3n, \ldots \qquad d = -3$

The sequence with nth term $a_n = \frac{1}{4}(n+3)$.

$$1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n+3}{4}, \ldots \qquad d = \frac{1}{4}$$

**Notice that the number being multiplied times n is the common difference.

The nth Term of an Arithmetic Sequence

The nth term of an arithmetic sequence has the form

$$a_n = dn + c$$

where d is the common difference between consecutive terms of the sequence, and $c = a_1 - d$.

An arithmetic sequence may be thought of as a linear function whose domain is the set of natural numbers.

$$y = mx + b \quad \leftrightarrow \quad a_n = dn + c$$

If graphed, the sequence would have slope d and y-intercept c.

<u>Alternative nth Term of an Arithmetic Sequence Formula</u>

Look at the pattern of an arithmetic sequence:

$$
\begin{aligned}
&a_1 = a_1 \\
&a_2 = a_1 + d \\
&a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d \\
&a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\
&a_5 = a_4 + d = (a_1 + 3d) + d = a_1 + 4d \\
&\vdots \\
&a_n = a_1 + (n-1)d
\end{aligned}
$$

1 less than the value of the index.

The nth term of an arithmetic sequence has the form

$$a_n = a_1 + (n\text{-}1)d$$

where d is the common difference between consecutive terms of the sequence.

Example: Find the *n*th term of the arithmetic sequence with common difference 5 and first term 9.

$$a_n = a_1 + (n-1)d$$
$$a_n = 9 + (n-1)5$$
$$a_n = 9 + 5n - 5$$
$$a_n = 5n + 4$$

<u>or</u>

$$a_n = dn + c$$
$$a_n = 5n + c$$
$$c = a_1 - d = 9 - 5 = 4, \text{ so}$$
$$a_n = 5n + 4$$

Example: Find a formula for the *n*th term of the arithmetic sequence whose common difference is -2 and whose first term is 7.

$$a_n = a_1 + (n-1)d$$
$$a_n = 7 + (n-1)(-2)$$
$$a_n = 7 - 2n + 2$$
$$a_n = -2n + 9$$

<u>or</u>

$$a_n = dn + c$$
$$a_n = -2n + c$$
$$c = a_1 - d = 7 - (-2) = 9, \text{ so}$$
$$a_n = -2n + 9$$

You can use either formula that you like.

Writing the Terms of an Arithmetic Sequence

You can find the terms of an arithmetic sequence given any 2 terms of the sequence.

Example: The 4th term of an arithmetic sequence is 20, and the 13th term is 65. Write the first several terms of this sequence.

We have $a_4 = 20$ and $a_{13} = 65$.

Remember that to get from one term to the next, we always add d, so to get from the 4th term to the 13th term, we just add 13 – 4 = 9 d's. In other words,

$$a_{13} = a_4 + 9d$$

If we substitute the values for a_{13} and a_4 we get

$$a_{13} = a_4 + 9d$$
$$65 = 20 + 9d$$
$$45 = 9d$$
$$d = 5$$

This leads us to the sequence

5, 10, 15, 20, 25, 30, 35, 40, …
$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots$$

Example: Find the *n*th term of the arithmetic sequence with fifth term 19 and ninth term 27.

Solution: To get from the 5th term to the 9th term, we need to start with the 5th term and add 9 – 5 = 4 d's.

$$a_9 = a_5 + 4d$$
$$27 = 19 + 4d$$
$$8 = 4d$$
$$d = 2$$

To find the *n*th term of the sequence, take the general formula and put in the values for one of the given terms in order to solve for a_1.

$$a_n = a_1 + (n-1)d$$
$$a_5 = a_1 + (5-1)(2)$$
$$19 = a_1 + (4)2$$
$$19 = a_1 + 8$$
$$a_1 = 11$$

That means that

$$a_n = a_1 + (n-1)d$$
$$a_n = 11 + (n-1)(2)$$
$$a_n = 11 + 2n - 2$$
$$a_n = 2n + 9$$

Recursion Formula

If you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the (n+1)th term by using the recursion formula

$$a_{n+1} = a_n + d$$

Example: Write the first 5 terms of the arithmetic sequence using the recursion formula if $a_1 = 6$ and $a_{k+1} = a_k + 5$.

Solution:

$$a_1 = 6$$
$$a_2 = a_1 + 5 = 6 + 5 = 11$$
$$a_3 = a_2 + 5 = 11 + 5 = 16$$
$$a_4 = a_3 + 5 = 16 + 5 = 21$$
$$a_5 = a_4 + 5 = 21 + 5 = 26$$

The Sum of a Finite Arithmetic Sequence

Problem: Add the integers from 1 to 100.

Solution: To find the sum, add the sequence twice and then divide the sum by 2.

$$\begin{array}{r}
1+\quad 2+\quad 3+\ldots+100 \\
\underline{100+99+98+\ldots+\quad\ 1} \\
101+101+101+\ldots+101
\end{array}$$

$$\frac{100\cdot 101}{2}=5050$$

The integers from 1 to 100 form an arithmetic sequence that has 100 terms. We have

$$a_1=1 \quad \text{and} \quad a_{100}=100$$

From that we can take the last line of the sum above, and we can write

$$\frac{100\cdot 101}{2}=\frac{100}{2}(100+1)$$

The general sum could be written as

$$\frac{100}{2}(a_1+a_{100})$$

Definition of the Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Example: Find the sum of the first 20 terms of the arithmetic sequence 2, 5, 8, 11, …

In order to use the formula, we will need the 20th term.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_{20} &= 2 + (20-1)3 \\ a_{20} &= 59 \end{aligned}$$

Now we can use the formula:

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ S_{20} &= \frac{20}{2}(2 + 59) \\ S_{20} &= 10(61) \\ S_{20} &= 610 \end{aligned}$$

Example: Evaluate $\sum_{n=1}^{12} 7n+1$.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{12} = \frac{12}{2}(8+85)$$

$$S_{12} = 6(93)$$

$$S_{12} = 558$$

Example: Find the sum of the first 10 terms of the arithmetic sequence 0.5, 0.9, 1.3, 1.7, …

$$a_n = a_1 + (n-1)d$$

$$a_{10} = 0.5 + (10-1)(0.4)$$

$$a_{10} = 4.1$$

Now we can use the formula:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{10} = \frac{10}{2}(0.5+4.1)$$

$$S_{10} = 23$$

Applications

Example: Each row of an auditorium has two more seats than the preceding row. Find the seating capacity of the auditorium if the front row has 30 seats and there are 40 rows.

Solution: What we are looking at is a sequence that starts with the number 30 and has 40 terms, with the common difference of 2. We need to find the sum of the sequence.

First we need the 40th term:

$$a_n = a_1 + (n-1)$$

$$a_{40} = 30 + (40-1)(2)$$

$$a_{40} = 108$$

Now find the sum.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{40} = \frac{40}{2}(30 + 108)$$

$$S_{40} = 20(138)$$

$$S_{40} = 2760$$

Example: Suppose you put $100 under your mattress at the end of the month. You continue to put money under your mattress each month and increase the amount by $5 each time. How much money is under your mattress after one year?

Solution: What we are looking at is a sequence that starts with the number 100 and has 12 terms, with the common difference of 5. We need to find the sum of the sequence.

First we need the 12th term:

$$a_n = a_1 + (n-1)$$
$$a_{12} = 100 + (12-1)(5)$$
$$a_{12} = 155$$

Now find the sum.

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{12} = \frac{12}{2}(100 + 155)$$
$$S_{12} = 6(255)$$
$$S_{12} = \$1530$$