The Binomial Theorem

Remember that a binomial has 2 terms. Look at the following binomial $(x + y)^n$ for several values on n.

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

What observations can you make?

1. In each expansion, there are n +1 terms.

2. In each expansion, x and y have symmetrical roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.

3. The sum of the powers of each term is n.

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called binomial coefficients. To find them, you can use the Binomial Theorem.

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x+y)^n = x^n + nx^{n-1}y + \ldots + {}_nC_r x^{n-r}y^r + \ldots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

***Note**: The value of r is 1 less than the number of the term. So, if $r = 7$, then it is the 8th term.

Example: Evaluate the following.

a) ${}_{10}C_5$

solution: ${}_{10}C_5 = \frac{10!}{(10-5)!5!} = \frac{10!}{5!5!} = 252$

b) $\binom{8}{2}$

solution: $\binom{8}{2} = \frac{8!}{(8-2)!2!} = \frac{8!}{6!2!} = 28$

c) ${}_{12}C_4$

solution: ${}_{12}C_4 = \frac{12!}{(12-4)!4!} = \frac{12!}{8!4!} = 495$

d) $\binom{7}{7}$

solution: $\binom{7}{7} = \frac{7!}{(7-7)!7!} = \frac{7!}{0!7!} = \frac{7!}{(1)7!} = 1$

e) ${}_{6}C_0$

solution: ${}_{6}C_0 = \frac{6!}{(6-0)!0!} = \frac{6!}{6!(1)} = 1$

Finding Binomial Coefficients on a Graphing Calculator

To find ${}_nC_r$ on your calculator, do the following:

1. Type in the value of n on your main screen.
2. Press [MATH] [PRB] [nCr]
3. Type in the value of r on your main screen.
4. Press [ENTER].

Example: Evaluate the following using the [nCr] feature on your graphing calculator.

a) ${}_{10}C_5$ *solution*: 252

b) $\binom{8}{2}$ *solution:* 28

c) ${}_7C_4$ *solution*: 35

d) ${}_7C_3$ *solution*: 35

*Note: The answers for c) and d) are the same because of the symmetric property of binomial coefficients. This will always be true when the 2 numbers for r add up to the number for n.

Remember our binomial expansion:

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Each coefficient shows up twice because of the symmetric property of binomial coefficients.

For example,

$$\binom{12}{5} = 792 \quad \text{and} \quad \binom{12}{7} = 792$$

Each of these represents a coefficient in a binomial expansion, so you would expect to have it show up twice.

<u>Pascal's Triangle</u>

The famous French mathematician Blaise Pascal created this triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

The first and last numbers of each line are 1, and the other numbers are formed by adding the 2 numbers immediately above the number.

Pascal noticed that the numbers in his triangle matched the numbers that are binomial expansion coefficients.

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

The top row is called the “zeroth row” because the exponent on $(x + y)$ is 0. Then we have the 1st row, 2nd row, etc. The nth row would be the expansion of $(x + y)^n$.

Example: Use the 7th row of Pascal's Triangle to find the binomial coefficients for $(x + y)^8$.

	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1
↓		↓		↓		↓		↓		↓		↓		↓		↓
${}_8C_0$		${}_8C_1$		${}_8C_2$		${}_8C_3$		${}_8C_4$		${}_8C_5$		${}_8C_6$		${}_8C_7$		${}_8C_8$

Example: Expand $(x + 2)^4$

- You know that the first term, x, starts with an exponent of 4 and then goes *down* by 1 for each term.
- You know that the 2nd term, 2, starts with an exponent of 0 and then goes *up* by 1 for each term.
- The exponents on both terms must add to 4.
- Using the 4th row of Pascal's Triangle, you know the binomial coefficients.

Solution: The 4th row of Pascal's Triangle is 1, 4, 6, 4, 1.

$$(x+2)^4 = 1x^4(2)^0 + 4x^3(2)^1 + 6x^2(2)^2 + 4x^1(2)^3 + 1x^0(2)^4$$

$$(x+2)^4 = 1x^4(1) + 4x^3(2) + 6x^2(4) + 4x^1(8) + (1)(1)(16)$$

$$(x+2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$$

Example: Expand $(3x + 4)^3$

- You know that the first term, $3x$, starts with an exponent of 3 and then goes *down* by 1 for each term.
- You know that the 2nd term, 4, starts with an exponent of 0 and then goes *up* by 1 for each term.
- The exponents on both terms must add to 3.
- Using the 3rd row of Pascal's Triangle, you know the binomial coefficients.

Solution: The 3rd row of Pascal's Triangle is 1, 3, 3, 1.

$$(3x+4)^3 = 1(3x)^3(4)^0 + 3(3x)^2(4)^1 + 3(3x)^1(4)^2 + +1(3x)^0(4)^3$$

$$(3x+4)^3 = 1(27x^3)(1) + 3(9x^2)(4) + 3(3x)(16) + +1(1)(64)$$

$$(3x+4)^3 = 27x^3 + 108x^2 + 144x + 64$$

Example: Expand $(2x + 3)^5$

- You know that the first term, $2x$, starts with an exponent of 5 and then goes *down* by 1 for each term.
- You know that the 2nd term, 3, starts with an exponent of 0 and then goes *up* by 1 for each term.
- The exponents on both terms must add to 5.
- Using the 5th row of Pascal's Triangle, you know the binomial coefficients.

Solution: The 5th row of Pascal's Triangle is 1,5,10,10,5,1.

$$(2x+3)^5 = (2x)^5(3)^0 + 5(2x)^4(3)^1 + 10(2x)^3(3)^2 + 10(2x)^2(3)^3 + 5(2x)^1(3)^4 + (2x)^0(3)^5$$
$$(2x+3)^5 = 32x^5(1) + 5(16x^4)(3) + 10(8x^3)(9) + 10(4x^2)(27) + 5(2x)(81) + (1)(243)$$
$$(2x+3)^5 = 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$$

Expanding Binomials with Differences

To expand binomials with differences rather than sums, the signs will alternate.

Example: Expand $(x - 5)^4$

$$(x-5)^4 = 1x^4(-5)^0 + 4x^3(-5)^1 + 6x^2(-5)^2 + 4x^1(-5)^3 + 1x^0(-5)^4$$
$$(x-5)^4 = 1x^4(1) + 4x^3(-5) + 6x^2(25) + 4x^1(-125) + (1)(1)(625)$$
$$(x-5)^4 = x^4 - 20x^3 + 150x^2 - 500x + 625$$

Example: Expand $(x^2 + 2)^3$

Solution: The 3rd row of Pascal's Triangle is 1, 3, 3, 1.

$$(x^2+2)^3 = 1(x^2)^3(2)^0 + 3(x^2)^2(2)^1 + 3(x^2)^1(2)^2 + +1(x^2)^0(2)^3$$
$$(x^2+2)^3 = 1(x^6)(1) + 3(x^4)(2) + 3(x^2)(4) + +1(1)(8)$$
$$(x^2+2)^3 = x^6 + 6x^4 + 12x^2 + 8$$

Example: Find the 6th term of $(k + 2m)^8$

<u>*Solution*</u>: Remember the 8th row of Pascal's Triangle from the beginning of this lesson.

	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1
↓		↓		↓		↓		↓		↓		↓		↓		↓
${}_8C_0$		${}_8C_1$		${}_8C_2$		${}_8C_3$		${}_8C_4$		${}_8C_5$		${}_8C_6$		${}_8C_7$		${}_8C_8$

The coefficient of each term is ${}_nC_r$. We know that $n = 8$. Since r starts with 0 for the first term, the 6th term will have $r = 5$. (r is always 1 less than the term position.)

So, ${}_8C_5 = 56$ is the coefficient.

The Binomial Theorem says that

$$(x+y)^n = x^n + nx^{n-1}y + ... + {}_nC_r x^{n-r} y^r + ... + nxy^{n-1} + y^n$$

We see that each term is of the form ${}_nC_r x^{n-r} y^r$.

We know that for our problem:
$n = 8$ (because our binomial is raised to 8th power),
$r = 5$ (because it is the 6th term that we are looking for),
$x = k$ (because the 1st term in our binomial is k), and
$y = 2m$ (because the 2nd term in our binomial is m).

$${}_nC_r x^{n-r} y^r = {}_8C_5 (k)^{8-5} (2m)^5 = 56k^3 32m^5$$

Example: Find the coefficient of the term a^6b^5 in the expansion of $(3a-2b)^{11}$.

Solution: We know a^6b^5 came from ${}_nC_r x^{n-r}y^r$. We already know that n = 11. This tells us that r must be 5. This means we have

$$\begin{aligned} {}_nC_r x^{n-r}y^r &= {}_{11}C_5(3a)^{11-5}(-2b)^5 \\ &= 462(3a)^6(-2b)^5 \\ &= -10{,}777{,}536a^6b^5 \end{aligned}$$

So, the coefficient is -10,777,536.