A.8 Graphical Representation of Data

▶ What you should learn

- How to plot points in the Cartesian plane
- How to use the Distance Formula to find the distance between two points
- How to use the Midpoint Formula to find the midpoint of a line segment
- How to use a coordinate plane to model and solve real-life problems

▶ Why you should learn it

The Cartesian plane can be used to represent relationships between two variables. For instance, in Exercise 27 on page A85, a graph represents the minimum wage in the United States from 1950 to 2000.

The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** named after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure A.15. The horizontal real number line is usually called the **x-axis,** and the vertical real number line is usually called the **y-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

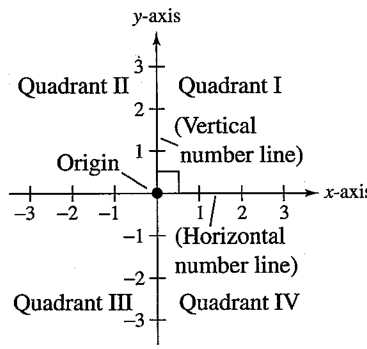


FIGURE A.15

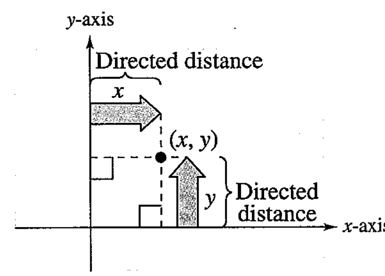
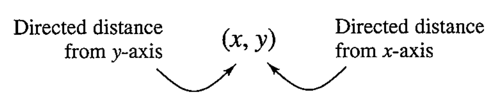


FIGURE A.16

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called **coordinates** of the point. The **x-coordinate** represents the directed distance from the y-axis to the point, and the **y-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure A.16.



The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

Example 1 ▶ Plotting Points in the Cartesian Plane

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

Solution

To plot the point $(-1, 2)$, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. The other four points can be plotted in a similar way, as shown in Figure A.17.

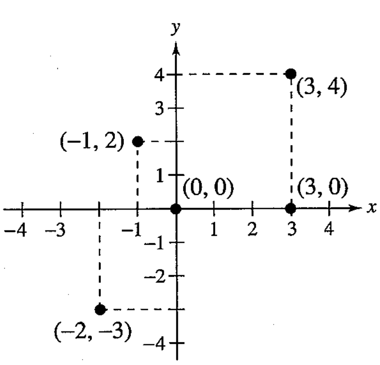


FIGURE A.17

The beauty of a rectangular coordinate system is that it allows you to *see* relationships between two variables. It would be difficult to overestimate the importance of Descartes's introduction of coordinates in the plane. Today, his ideas are in common use in virtually every scientific and business-related field.

Example 2 ▶ Sketching a Scatter Plot

Year, t	Amount, A
1990	475
1991	577
1992	521
1993	569
1994	609
1995	562
1996	707
1997	723
1998	718
1999	739

From 1990 through 1999, the amount A (in millions of dollars) spent on skiing equipment in the United States is shown in the table, where t represents the year. Sketch a scatter plot of the data. (Source: National Sporting Goods Association)

Solution

To sketch a *scatter plot* of the data shown in the table, you simply represent each pair of values by an ordered pair (t, A) and plot the resulting points, as shown in Figure A.18. For instance, the first pair of values is represented by the ordered pair (1990, 475). Note that the break in the t-axis indicates that the numbers between 0 and 1990 have been omitted.

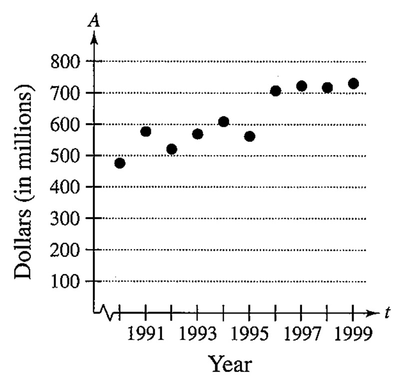


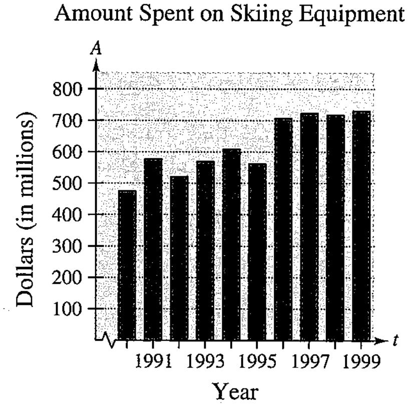
FIGURE A.18

STUDY TIP

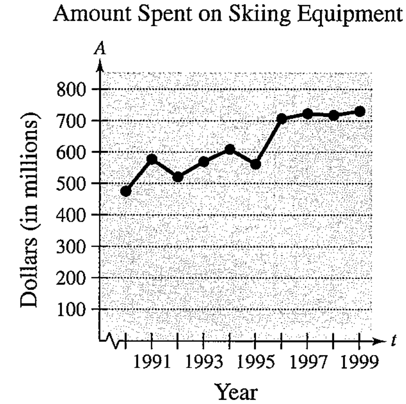
In Example 2, you could have let $t = 1$ represent the year 1990. In that case, the horizontal axis would not have been broken, and the tick marks would have been labeled 1 through 10 (instead of 1990 through 1999).

Technology

The scatter plot in Example 2 is only one way to represent the data graphically. Two other techniques are shown at the right. The first is a bar graph and the second is a line graph. All three graphical representations were created with a computer. If you have access to a graphing utility, try using it to represent graphically the data given in Example 2.



Bar Graph



Line Graph

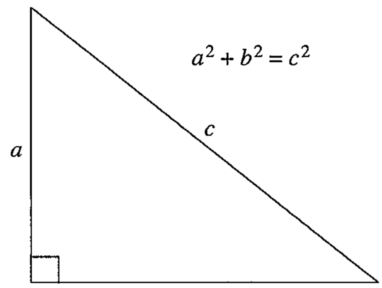


FIGURE A.19

The Distance Formula

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and sides of lengths a and b, you have

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

as shown in Figure A.19. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. With these two points, a right triangle can be formed, as shown in Figure A.20. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, you can write

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This result is the **Distance Formula.**

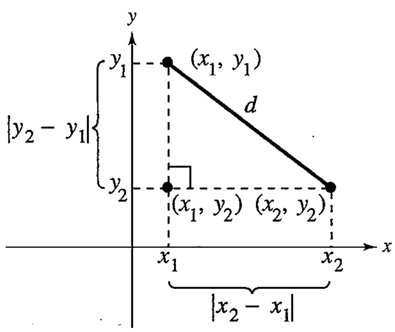


FIGURE A.20

> **The Distance Formula**
>
> The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 3 ▶ **Finding a Distance**

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Solution

Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[3 - (-2)]^2 + (4 - 1)^2}$	Substitute for x_1, y_1, x_2, and y_2.
$= \sqrt{(5)^2 + (3)^2}$	Simplify.
$= \sqrt{34}$	Simplify.
≈ 5.83	Use a calculator.

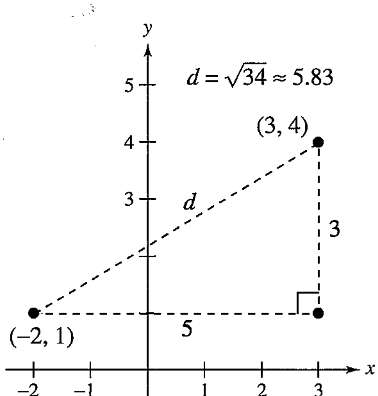


FIGURE A.21

Note in Figure A.21 that a distance of 5.83 looks about right. You can use the Pythagorean Theorem to check that the distance is correct.

$d^2 \stackrel{?}{=} 3^2 + 5^2$	Pythagorean Theorem
$(\sqrt{34})^2 \stackrel{?}{=} 3^2 + 5^2$	Substitute for d.
$34 = 34$	Distance checks. ✓

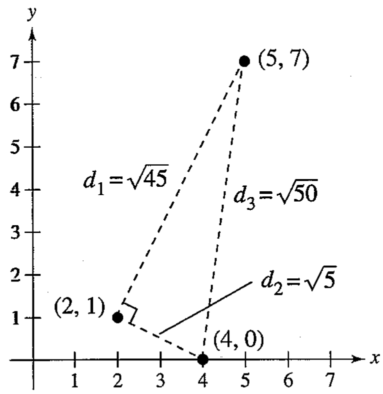


FIGURE A.22

Example 4 ▶ Verifying a Right Triangle

Show that the points (2, 1), (4, 0), and (5, 7) are vertices of a right triangle.

Solution

The three points are plotted in Figure A.22. Using the Distance Formula, you can find the lengths of the three sides as follows.

$$d_1 = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45}$$

$$d_2 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$d_3 = \sqrt{(5-4)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50}$$

Because

$$(d_1)^2 + (d_2)^2 = 45 + 5$$
$$= 50$$
$$= (d_3)^2$$

you can conclude that the triangle must be a right triangle.

The figures provided with Examples 3 and 4 were not really essential to the solution. Nevertheless, it is strongly recommended that you develop the habit of including sketches with your solutions—even if they are not required.

Example 5 ▶ Finding the Length of a Pass

During the first quarter of the 2002 Orange Bowl, Brock Berlin, the quarterback for the University of Florida, threw a pass from the 37-yard line, 40 yards from the sideline. The pass was caught by the wide receiver Taylor Jacobs on the 3-yard line, 20 yards from the same sideline, as shown in Figure A.23. How long was the pass?

Solution

You can find the length of the pass by finding the distance between the points (40, 37) and (20, 3).

$$d = \sqrt{(40-20)^2 + (37-3)^2} \quad \text{Distance Formula}$$

$$= \sqrt{400 + 1156} \quad \text{Simplify.}$$

$$= \sqrt{1556} \quad \text{Simplify.}$$

$$\approx 39 \quad \text{Use a calculator.}$$

So, the pass was about 39 yards long.

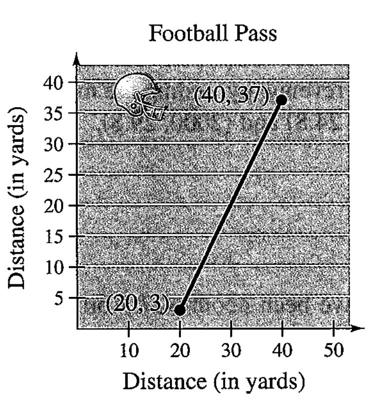


FIGURE A.23

In Example 5, the scale along the goal line does not normally appear on a football field. However, when you use coordinate geometry to solve real-life problems, you are free to place the coordinate system in any way that is convenient for the solution of the problem.

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can simply find the average values of the respective coordinates of the two endpoints using the **Midpoint Formula.**

> **The Midpoint Formula**
>
> The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by the Midpoint Formula
>
> $$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 6 ▶ Finding a Line Segment's Midpoint

Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$, as shown in Figure A.24.

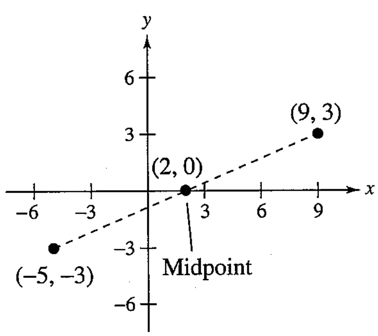


FIGURE A.24

Solution

Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \text{Midpoint Formula}$$

$$= \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2.$$

$$= (2, 0) \qquad \text{Simplify.}$$

Example 7 ▶ Estimating Annual Revenue

The United Parcel Service had annual revenues of \$24.8 billion in 1998 and \$29.8 billion in 2000. Without knowing any additional information, what would you estimate the 1999 revenue to have been? (Source: United Parcel Service of America Corp.)

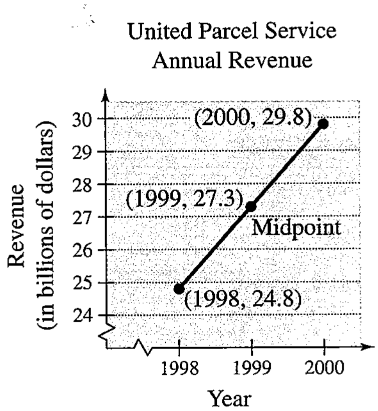


FIGURE A.25

Solution

One solution to the problem is to assume that revenue followed a linear pattern. With this assumption, you can estimate the 1999 revenue by finding the midpoint of the line segment connecting the points (1998, 24.8) and (2000, 29.8).

$$\text{Midpoint} = \left(\frac{1998 + 2000}{2}, \frac{24.8 + 29.8}{2}\right)$$

$$= (1999, 27.3)$$

So, you would estimate the 1999 revenue to have been about \$27.3 billion, as shown in Figure A.25. (The actual 1999 revenue was \$27.1 billion.)

Application

Example 8 ▶ Translating Points in the Plane

The triangle in Figure A.26 has vertices at the points $(-1, 2)$, $(1, -4)$, and $(2, 3)$. Shift the triangle three units to the right and two units upward and find the vertices of the shifted triangle, as shown in Figure A.27.

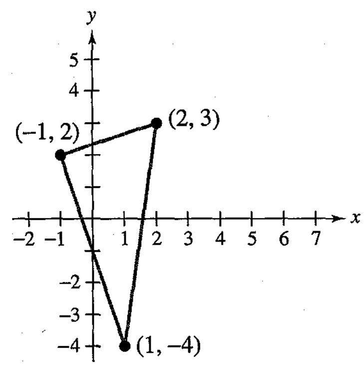


FIGURE A.26

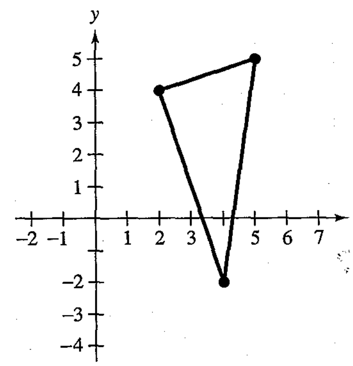


FIGURE A.27

Solution

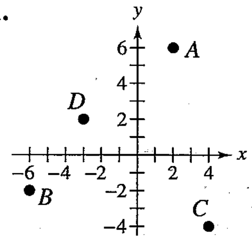
To shift the vertices three units to the right, add 3 to each of the x-coordinates. To shift the vertices two units upward, add 2 to each of the y-coordinates.

Original Point	*Translated Point*
$(-1, 2)$	$(-1 + 3, 2 + 2) = (2, 4)$
$(1, -4)$	$(1 + 3, -4 + 2) = (4, -2)$
$(2, 3)$	$(2 + 3, 3 + 2) = (5, 5)$

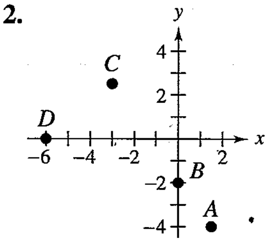
A.8 Exercises

In Exercises 1 and 2, approximate the coordinates of the points.

1.



2.



In Exercises 3–6, find the coordinates of the point.

3. The point is located three units to the left of the y-axis and four units above the x-axis.

4. The point is located eight units below the x-axis and four units to the right of the y-axis.

5. The point is located five units below the x-axis and the coordinates of the point are equal.

6. The point is on the x-axis and 12 units to the left of the y-axis.

In Exercises 7–16, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

7. $x > 0$ and $y < 0$

8. $x < 0$ and $y < 0$

9. $x = -4$ and $y > 0$

10. $x > 2$ and $y = 3$

11. $y < -5$

12. $x > 4$

13. $(x, -y)$ is in the second quadrant.

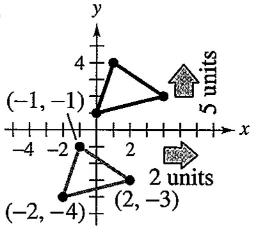
14. $(-x, y)$ is in the fourth quadrant.

15. $xy > 0$

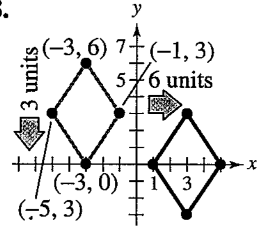
16. $xy < 0$

In Exercises 17–20, the polygon is shifted to a new position in the plane. Find the coordinates of the vertices of the polygon in its new position.

17.



18.



19. Original coordinates of vertices:

$(-7, -2), (-2, 2), (-2, -4), (-7, -4)$

Shift: eight units upward, four units to the right

20. Original coordinates of vertices:

$(5, 8), (3, 6), (7, 6), (5, 2)$

Shift: 6 units downward, 10 units to the left

In Exercises 21 and 22, sketch a scatter plot of the data given in the table.

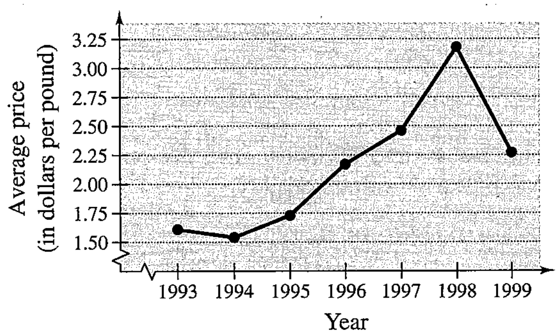
21. ***Meteorology*** The table shows the lowest temperature on record y (in degrees Fahrenheit) in Duluth, Minnesota, for each month x, where $x = 1$ represents January. (Source: NOAA)

Month, x	Temperature, y
1	−39
2	−33
3	−29
4	−5
5	17
6	27
7	35
8	32
9	22
10	8
11	−23
12	−34

22. ***Number of Stores*** The table shows the number y of Wal-Mart stores for each year x from 1993 through 2000. (Source: Wal-Mart Stores, Inc.)

Year, x	Number of stores, y
1993	2440
1994	2759
1995	2943
1996	3054
1997	3406
1998	3599
1999	3985
2000	4190

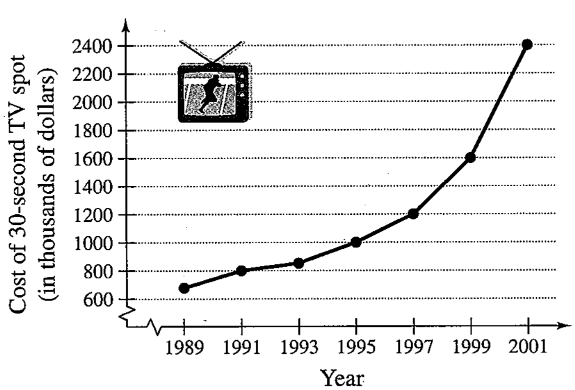
Retail Price **In Exercises 23 and 24, use the graph below, which shows the average retail price of 1 pound of butter from 1993 to 1999.** (Source: U.S. Bureau of Labor Statistics)



23. Approximate the highest price of a pound of butter shown in the graph. When did this occur?

24. Approximate the percent change in the price of butter from the price in 1994 to the highest price shown in the graph.

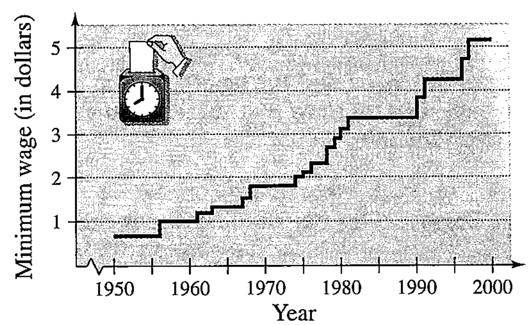
Advertising **In Exercises 25 and 26, use the graph below, which shows the cost of a 30-second television spot (in thousands of dollars) during the Super Bowl from 1989 to 2001.** (Source: USA Today Research)



25. Approximate the percent increase in the cost of a 30-second spot from Super Bowl XXIII in 1989 to Super Bowl XXXV in 2001.

26. Estimate the percent increase in the cost of a 30-second spot (a) from Super Bowl XXIII in 1989 to Super Bowl XXVII in 1993 and (b) from Super Bowl XXVII in 1993 to Super Bowl XXXV in 2001.

27. ***Labor Force*** Use the graph below, which shows the minimum wage in the United States (in dollars) from 1950 to 2000. (Source: U.S. Employment Standards Administration)



(a) Which decade shows the greatest increase in minimum wage?

(b) Approximate the percent increases in the minimum wage from 1990 to 1995 and from 1995 to 2000.

(c) Use the percent increase from 1995 to 2000 to predict the minimum wage in 2005.

(d) Do you believe that your prediction in part (c) is reasonable? Explain.

28. ***Data Analysis*** Use the table below, which shows the mathematics entrance test scores x and the final examination scores y in an algebra course for a sample of 10 students.

x	22	29	35	40	44
y	53	74	57	66	79

x	48	53	58	65	76
y	90	76	93	83	99

(a) Sketch a scatter plot of the data shown in the table.

(b) Find the entrance exam score of any student with a final exam score in the 80s.

(c) Does a higher entrance exam score imply a higher final exam score? Explain.

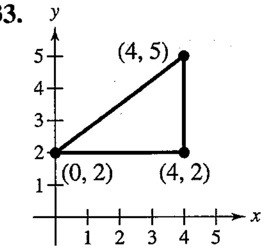
In Exercises 29–32, find the distance between the points. (*Note:* In each case, the two points lie on the same horizontal or vertical line.)

29. $(6, -3), (6, 5)$ **30.** $(1, 4), (8, 4)$

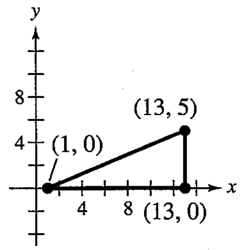
31. $(-3, -1), (2, -1)$ **32.** $(-3, -4), (-3, 6)$

In Exercises 33–36, (a) find the length of each side of the right triangle, and (b) show that these lengths satisfy the Pythagorean Theorem.

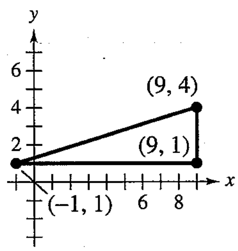
33.



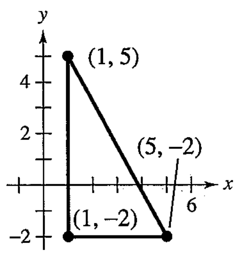
34.



35.



36.



In Exercises 37–46, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

37. $(1, 1), (9, 7)$ **38.** $(1, 12), (6, 0)$

39. $(-4, 10), (4, -5)$ **40.** $(-7, -4), (2, 8)$

41. $(-1, 2), (5, 4)$ **42.** $(2, 10), (10, 2)$

43. $\left(\frac{1}{2}, 1\right), \left(-\frac{5}{2}, \frac{4}{3}\right)$ **44.** $\left(-\frac{1}{3}, -\frac{1}{3}\right), \left(-\frac{1}{6}, -\frac{1}{2}\right)$

45. $(6.2, 5.4), (-3.7, 1.8)$

46. $(-16.8, 12.3), (5.6, 4.9)$

Sales **In Exercises 47 and 48, use the Midpoint Formula to estimate the sales of Target Corporation and Kmart Corporation in 1998, given the sales in 1996 and 2000. Assume that the sales followed a linear pattern.**

47. *Target*

Year	Sales (in millions)
1996	\$25,371
2000	\$36,903

(Source: Target Corporation)

48. *Kmart*

Year	Sales (in millions)
1996	\$31,437
2000	\$37,028

(Source: Kmart Corporation)

In Exercises 49 and 50, show that the points form the vertices of the indicated polygon.

49. Right triangle: $(4, 0), (2, 1), (-1, -5)$

50. Isosceles triangle: $(1, -3), (3, 2), (-2, 4)$

51. A line segment has (x_1, y_1) as one endpoint and (x_m, y_m) as its midpoint. Find the other endpoint (x_2, y_2) of the line segment in terms of $x_1, y_1, x_m,$ and y_m.

52. Use the result of Exercise 51 to find the coordinates of the endpoint of a line segment if the coordinates of the other endpoint and midpoint are, respectively,

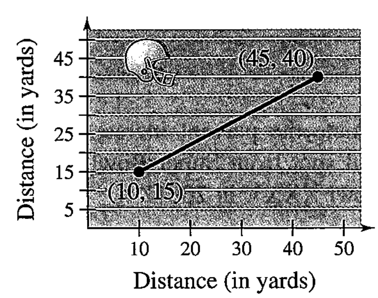
(a) $(1, -2), (4, -1)$ and (b) $(-5, 11), (2, 4)$.

53. Use the Midpoint Formula three times to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four parts.

54. Use the result of Exercise 53 to find the points that divide the line segment joining the given points into four equal parts.

(a) $(1, -2), (4, -1)$ (b) $(-2, -3), (0, 0)$

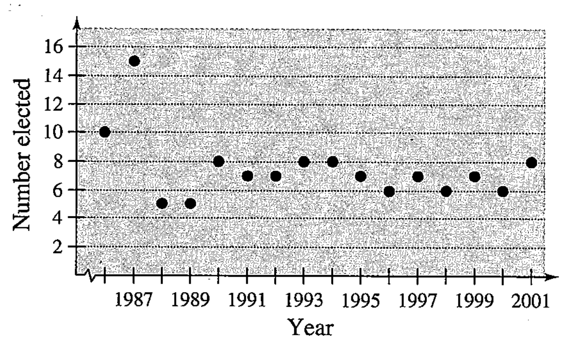
55. ***Sports*** In a football game, a quarterback throws a pass from the 15-yard line, 10 yards from the sideline, as shown in the figure. The pass is caught on the 40-yard line, 45 yards from the same sideline. How long is the pass?



56. ***Flying Distance*** A jet plane flies from Naples, Italy in a straight line to Rome, Italy, which is 120 kilometers west and 150 kilometers north of Naples. How far does the plane fly?

57. ***Make a Conjecture*** Plot the points $(2, 1)$, $(-3, 5)$, and $(7, -3)$ on a rectangular coordinate system. Then change the sign of the x-coordinate of each point and plot the three new points on the same rectangular coordinate system. Make a conjecture about the location of a point when each of the following occurs.
   (a) The sign of the x-coordinate is changed.
   (b) The sign of the y-coordinate is changed.
   (c) The signs of both the x- and y-coordinates are changed.

58. ***Music*** The graph shows the numbers of recording artists who were elected to the Rock and Roll Hall of Fame from 1986 to 2001.



   (a) Describe any trends in the data. From these trends, predict the number of artists elected in 2004.
   (b) Why do you think the numbers elected in 1986 and 1987 were greater than in other years?

59. ***Revenue*** Polo Ralph Lauren Corp. had annual revenues of \$1713.1 million in 1998 and \$2225.8 million in 2000. Use the Midpoint Formula to estimate the revenue in 1999. (Source: Polo Ralph Lauren Corp.)

60. ***Revenue*** Zale Corp. had annual revenues of \$1428.9 million in 1999 and \$2068.2 million in 2001. Use the Midpoint Formula to estimate the revenue in 2000. (Source: Zale Corp.)

Synthesis

True or False? **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

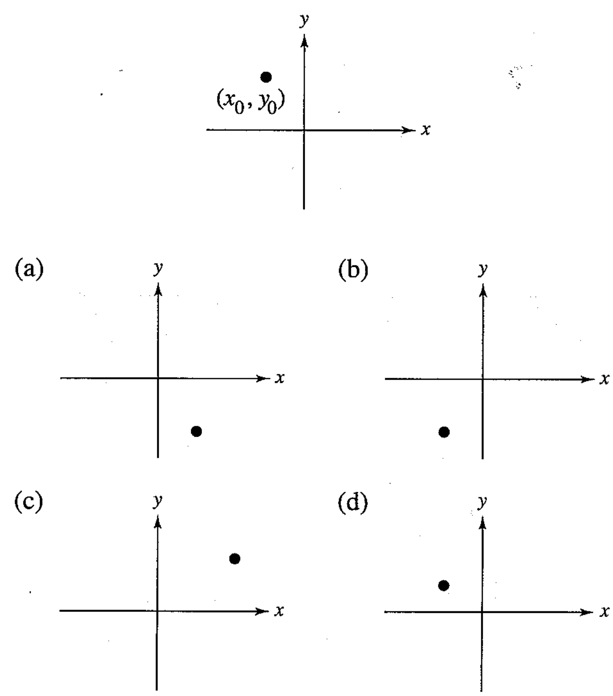
61. In order to divide a line segment into 16 equal parts, you would have to use the Midpoint Formula 16 times.

62. The points $(-8, 4)$, $(2, 11)$, and $(-5, 1)$ represent the vertices of an isosceles triangle.

63. ***Think About It*** What is the y-coordinate of any point on the x-axis? What is the x-coordinate of any point on the y-axis?

64. ***Think About It*** When plotting points on the rectangular coordinate system, is it true that the scales on the x- and y-axes must be the same? Explain.

In Exercises 65–68, use the plot of the point (x_0, y_0) in the figure. Match the transformation of the point with the correct plot. [The plots are labeled (a), (b), (c), and (d).]



65. $(x_0, -y_0)$

66. $(-2x_0, y_0)$

67. $\left(x_0, \frac{1}{2}y_0\right)$

68. $(-x_0, -y_0)$

69. ***Proof*** Prove that the diagonals of the parallelogram in the figure intersect at their midpoints.