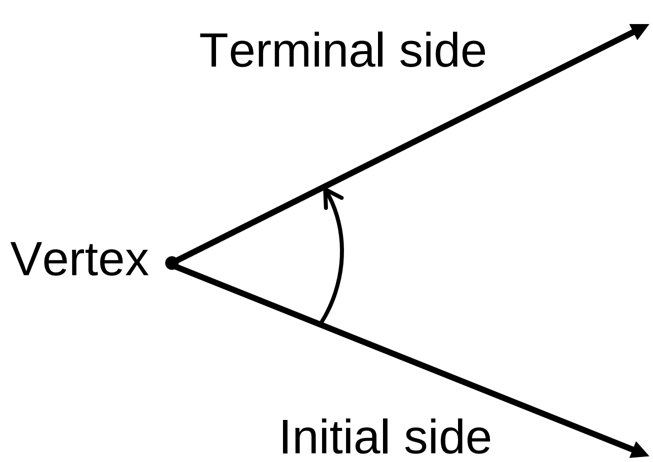
Trigonometry in the Cartesian Plane

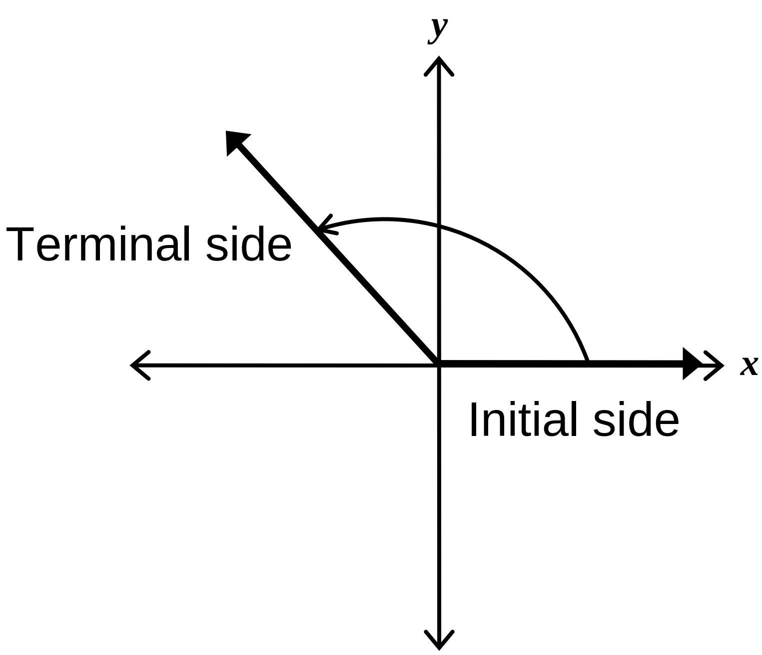
*__Trigonometry__ comes from the Greek word meaning "*measurement of triangles*." It primarily dealt with angles and triangles as it pertained to navigation, astronomy, and surveying. Today, the use has expanded to involve rotations, orbits, waves, vibrations, etc.

Definitions:

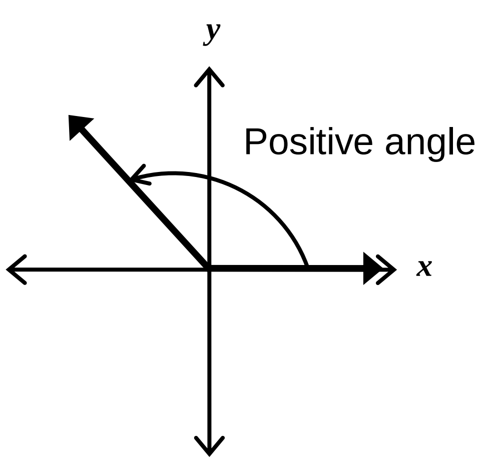
- An angle is determined by rotating a ray (half-line) about its endpoint.
- The initial side of an angle is the starting position of the rotated ray in the formation of an angle.
- The terminal side of an angle is the position of the ray after the rotation when an angle is formed.
- The vertex of an angle is the endpoint of the ray used in the formation of an angle.

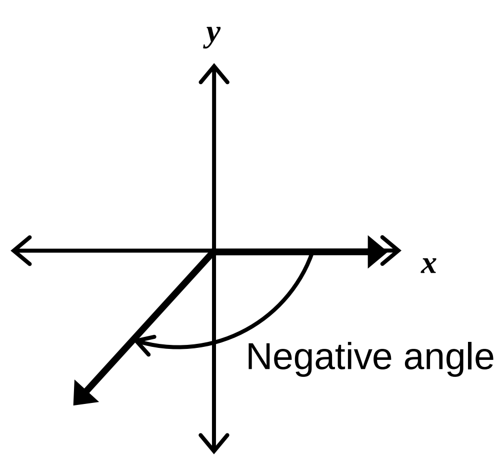


- An angle is in <u>standard position</u> when the angle's vertex is at the origin of a coordinate system and its initial side coincides with the positive *x*-axis.

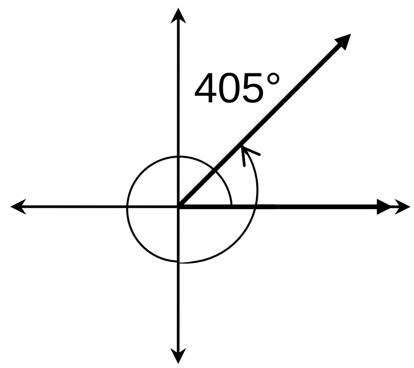


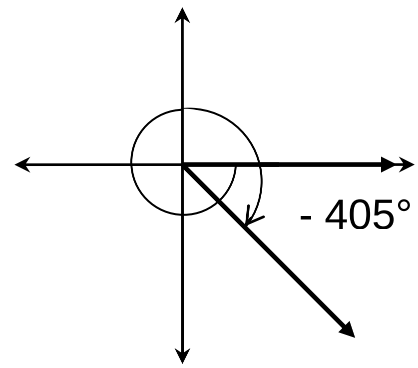
- A <u>positive angle</u> is generated by a counterclockwise rotation; whereas a <u>negative angle</u> is generated by a clockwise rotation.



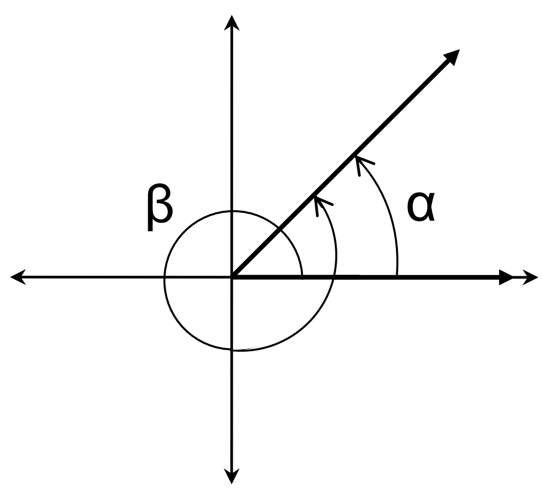


- An angle can be the result of more than one full rotation, in either the positive or negative direction.



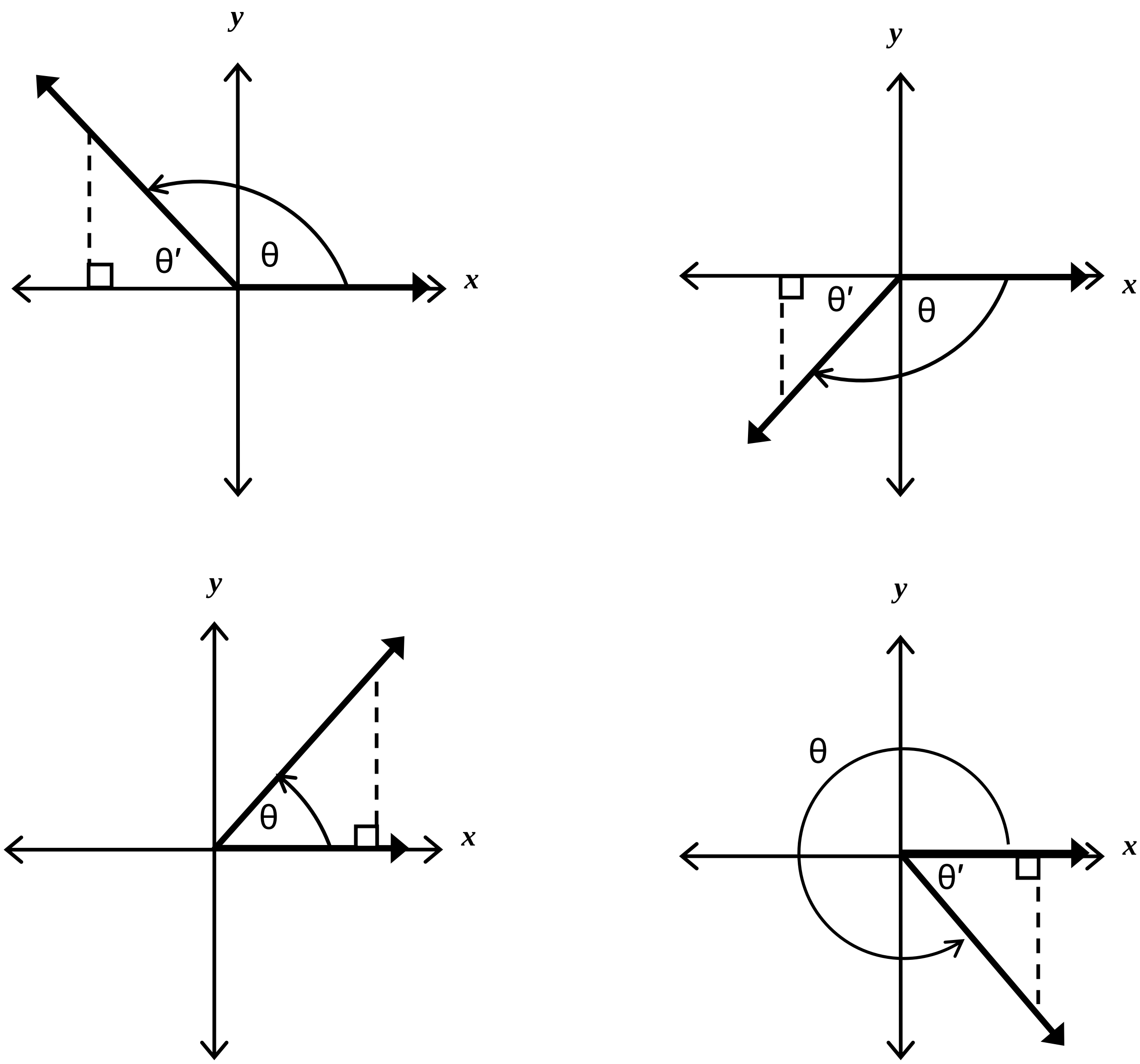


- If two angles are coterminal, then they have the same initial side and the same terminal side.



- The <u>reference angle</u> for an angle in standard position is the acute angle that the terminal side makes with the x-axis.
- The <u>reference triangle</u> is the right triangle formed which includes the reference angle.

The reference angle is θ′.



Example: Find the reference angle for the following angles.

a) $\theta = 125°$

answer: $\theta' = 180° - 125° = 55°$

b) $\theta = 330°$

answer: $\theta' = 360° - 330° = 30°$

c) $\theta = -230°$

answer: $\theta' = 230° - 180° = 50°$

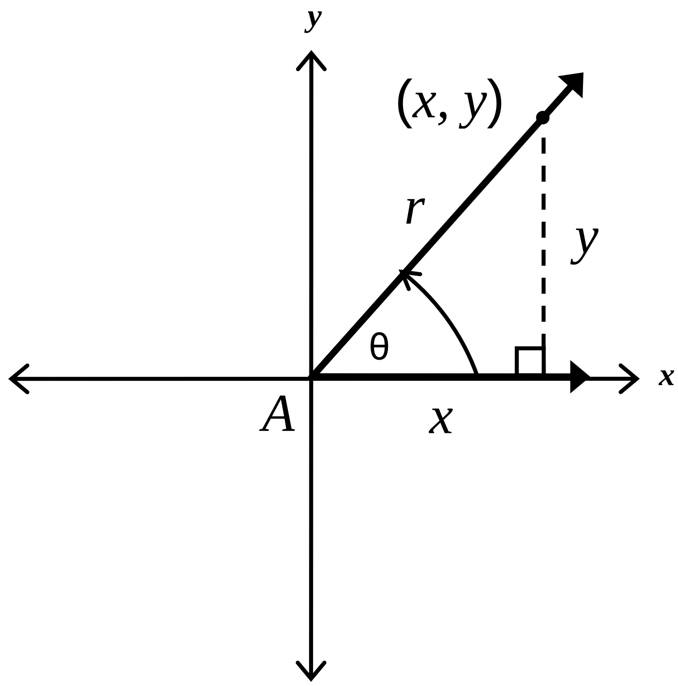
d) $\theta = 220°$

answer: $\theta' = 220° - 180° = 40°$

e) $\theta = -100°$

answer: $\theta' = 180° - 100° = 80°$

Definition of Trig Values for Acute Angles



$$\sin A = \frac{opp}{hyp} = \frac{y}{r} \qquad \csc A = \frac{r}{y}$$

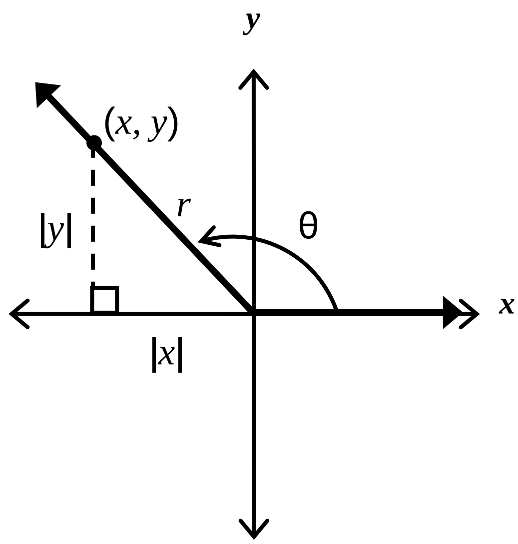
$$\cos A = \frac{adj}{hyp} = \frac{x}{r} \qquad \sec A = \frac{r}{x}$$

$$\tan A = \frac{opp}{adj} = \frac{y}{x} \qquad \cot A = \frac{x}{y}$$

with x, y, and $r \neq 0$.

Definition of Trig Values of Any Angle

Let θ be any angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$. Then

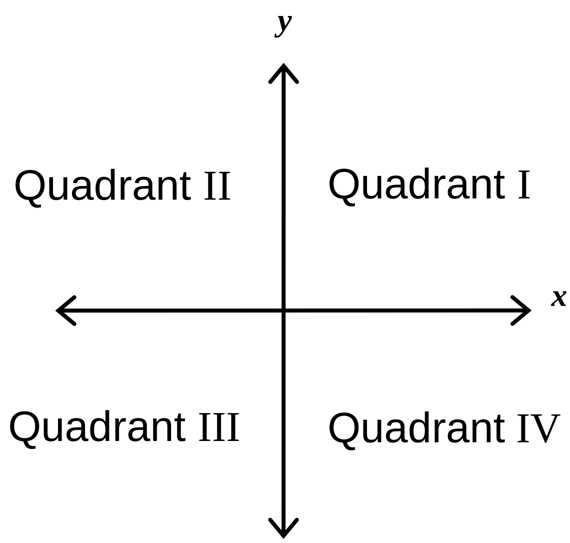


$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}, \quad y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}, \quad x \neq 0$$

$$\tan\theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

***Note**: The value of r is always positive, but the signs on x and y depend on the point (x, y), which will change depending on which quadrant (x, y) is in.

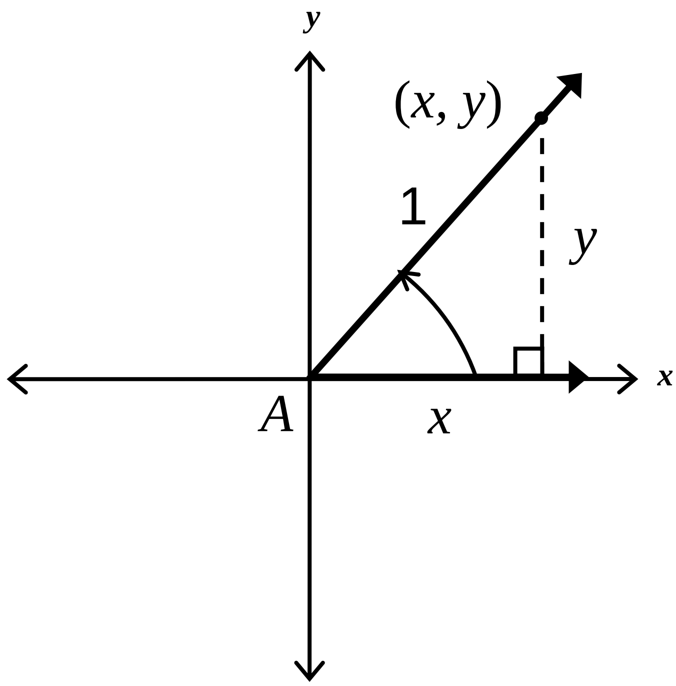


Quad I (+,+)	Quad II (-,+)	Quad III (-,-)	Quad IV (+,-)
$\sin A = \frac{y}{r} = \frac{+}{+} = +$	$\sin A = \frac{y}{r} = \frac{+}{+} = +$	$\sin A = \frac{y}{r} = \frac{-}{+} = -$	$\sin A = \frac{y}{r} = \frac{-}{+} = -$
$\cos A = \frac{x}{r} = \frac{+}{+} = +$	$\cos A = \frac{x}{r} = \frac{-}{+} = -$	$\cos A = \frac{x}{r} = \frac{-}{+} = -$	$\cos A = \frac{x}{r} = \frac{+}{+} = +$
$\tan A = \frac{y}{x} = \frac{+}{+} = +$	$\tan A = \frac{y}{x} = \frac{+}{-} = -$	$\tan A = \frac{y}{x} = \frac{-}{-} = +$	$\tan A = \frac{y}{x} = \frac{-}{+} = -$
$\csc A = \frac{r}{y} = \frac{+}{+} = +$	$\csc A = \frac{r}{y} = \frac{+}{+} = +$	$\csc A = \frac{r}{y} = \frac{+}{-} = -$	$\csc A = \frac{r}{y} = \frac{+}{-} = -$
$\sec A = \frac{r}{x} = \frac{+}{+} = +$	$\sec A = \frac{r}{x} = \frac{+}{-} = -$	$\sec A = \frac{r}{x} = \frac{+}{-} = -$	$\sec A = \frac{r}{x} = \frac{+}{+} = +$
$\cot A = \frac{x}{y} = \frac{+}{+} = +$	$\cot A = \frac{x}{y} = \frac{-}{+} = -$	$\cot A = \frac{x}{y} = \frac{-}{-} = +$	$\cot A = \frac{x}{y} = \frac{+}{-} = -$

allsintancos!!!!

(all)(sin)(tan)(cos) → What functions are positive, starting with Quadrant I.

Let's look at what the reference triangles look like when we choose (x, y) so that r=1.



Now we have:

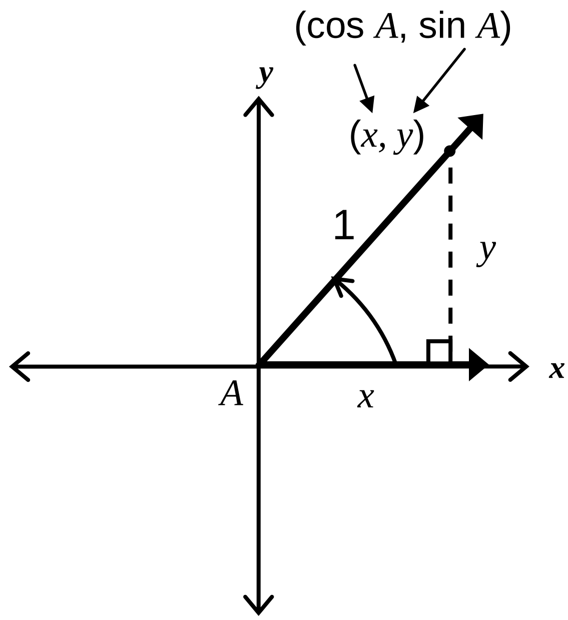
$$\sin A = \frac{y}{r} = \frac{y}{1} = y \qquad \csc A = \frac{1}{y}$$

$$\cos A = \frac{x}{r} = \frac{x}{1} = x \qquad \sec A = \frac{1}{x}$$

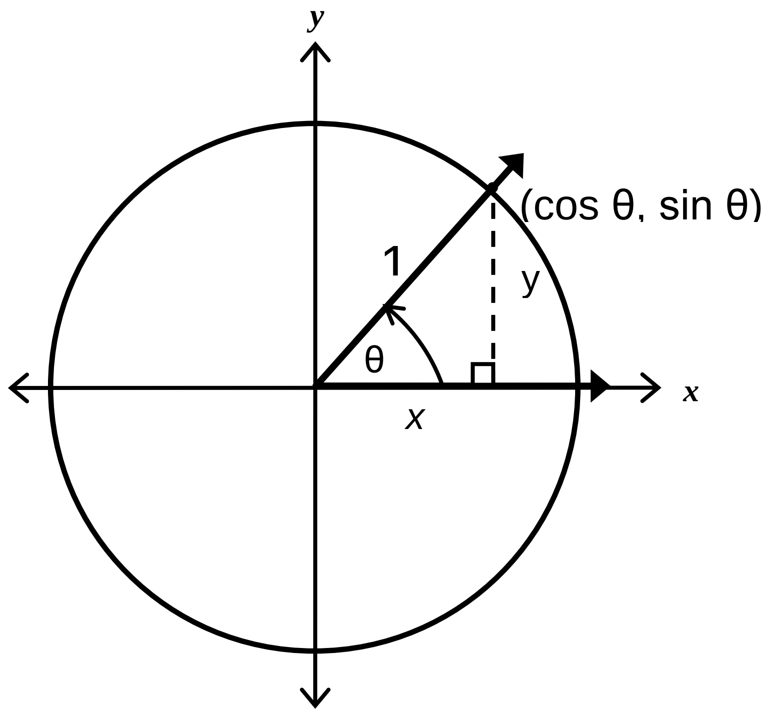
$$\tan A = \frac{y}{x} \qquad \cot A = \frac{x}{y}$$

**As long as $r = 1$, we have: $\cos A = x$
$\sin A = y$

The point (x,y) can be replaced with $(\cos A, \sin A)$ because $x=\cos A$ and $y=\sin A$.



Look at a circle with radius = 1.

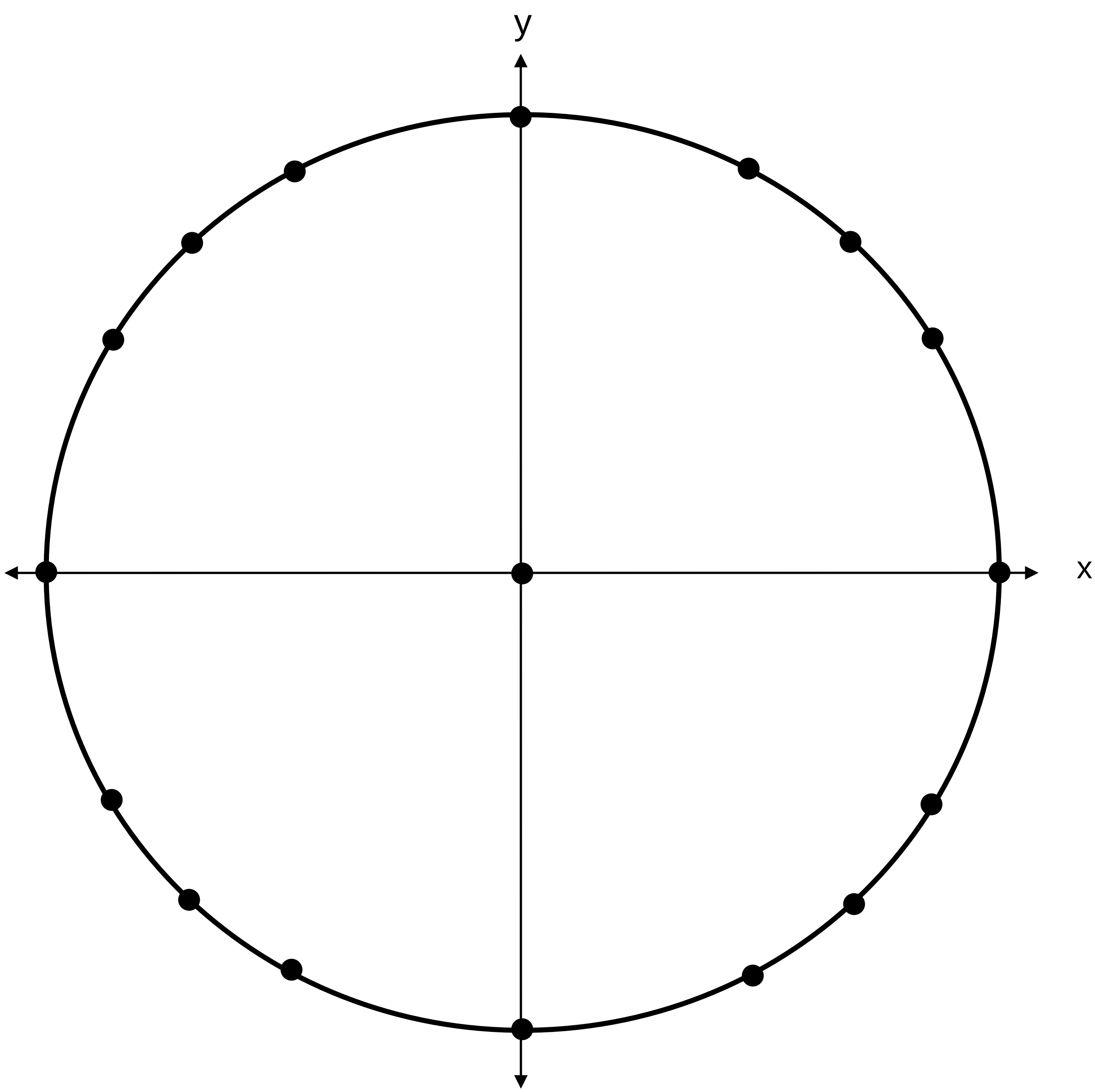


**For any coordinates on the unit circle, its coordinates are (cos θ, sin θ) where θ is an angle in standard position.

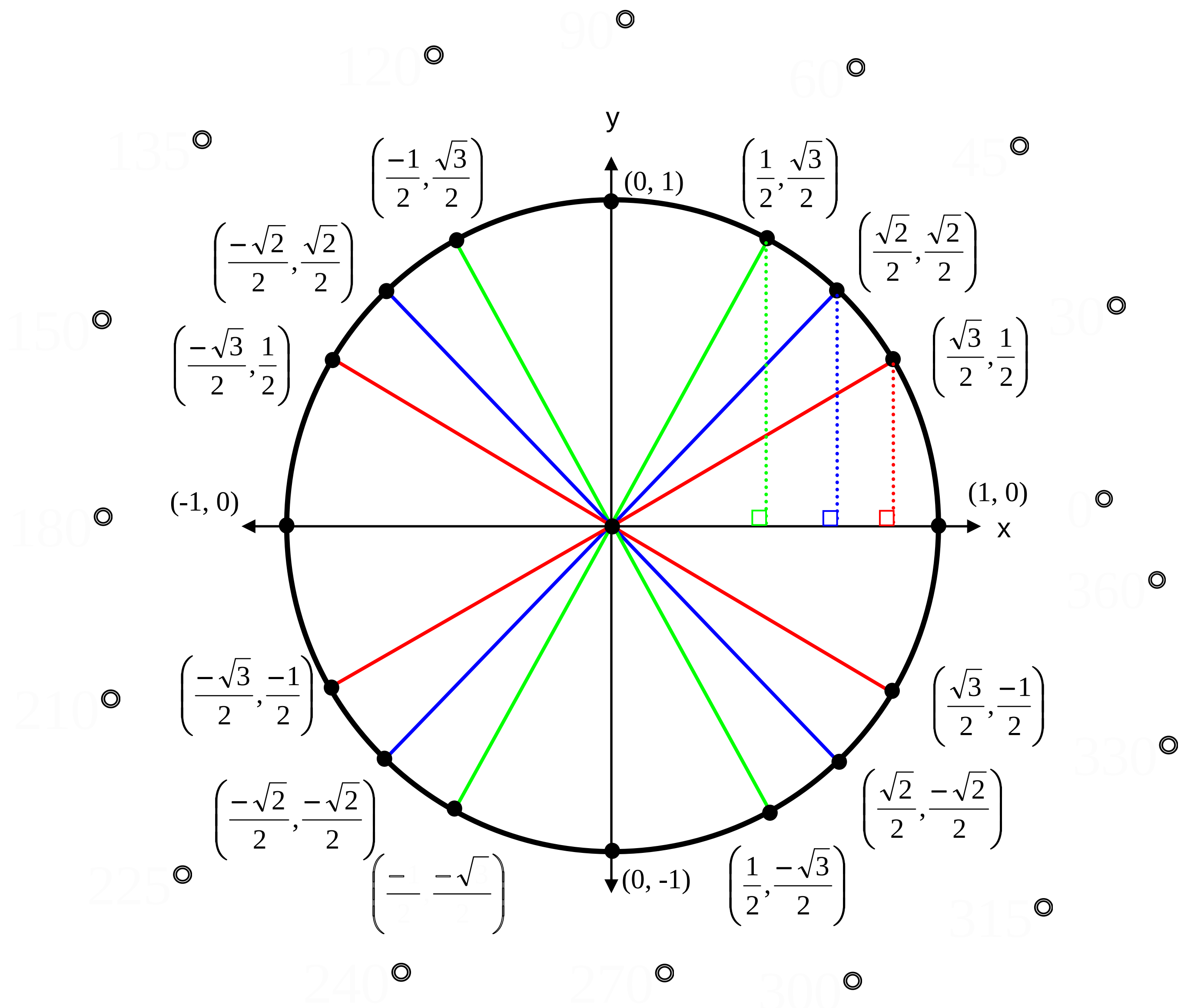
**If θ is <u>not</u> an acute angle, then we find the coordinates (x,y) (ie. cos θ, sin θ) by using the reference triangle.

Now take the unit circle and look at the common angles, their various reference triangles, and their trig values.

The Unit Circle



The Unit Circle



The cosine of angle θ is the x-coordinate.
The sine of angle θ is the y-coordinate.

$(x,y) = (\cos \theta, \sin \theta)$

Finding Trig Values with a Calculator

Example: Find the following.

a) csc 220°

- *Find sin 220° first.*
- *For sin 220° use the reference angle and find sin 40°.*
- *Since 220° is in quadrant III, sin 220° is negative*
- *So sin 220° = -.6428*
- *Then csc 220° = 1/sin 220° = -1.5557*

b) cot(-230°)

- *Find tan(-230°) first.*
- *For tan(-230°) use the reference angle and find tan 50°.*
- *Since -230° is in quadrant II, tan(-230°) is negative.*
- *So tan(-230°) = -1.1918*
- *Then cot (-230°) = 1/tan(-230°) = -.8391.*

c) cos 330°

- *The reference angle is 30°.*
- *Since 330° is in quadrant IV, cos 330° is positive.*
- *Then cos 330° = .8660*

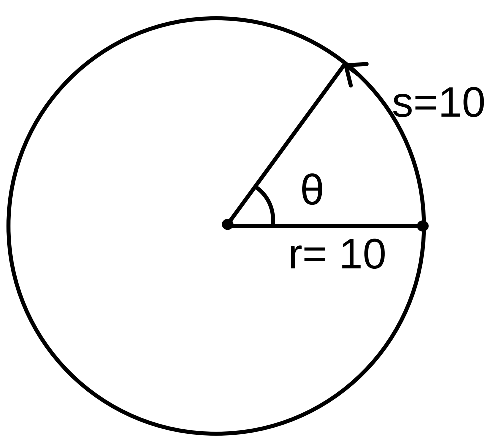
Radian Measure

Definitions:

- The measure of an angle is determined by the amount of rotation from the initial side to the terminal side.

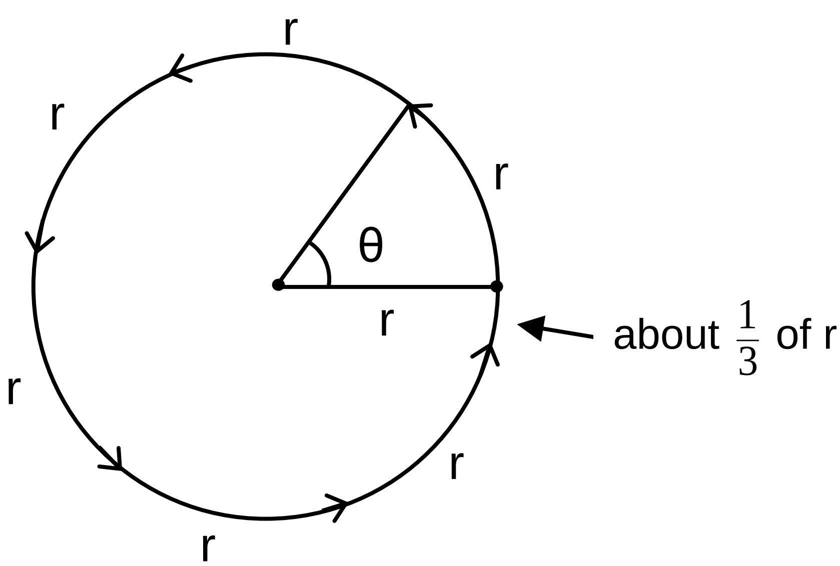
- A central angle is one whose vertex is the center of a circle.

- One radian is the measure of a central angle θ that intercepts an arc *s* equal in length to the radius *r* of the circle.

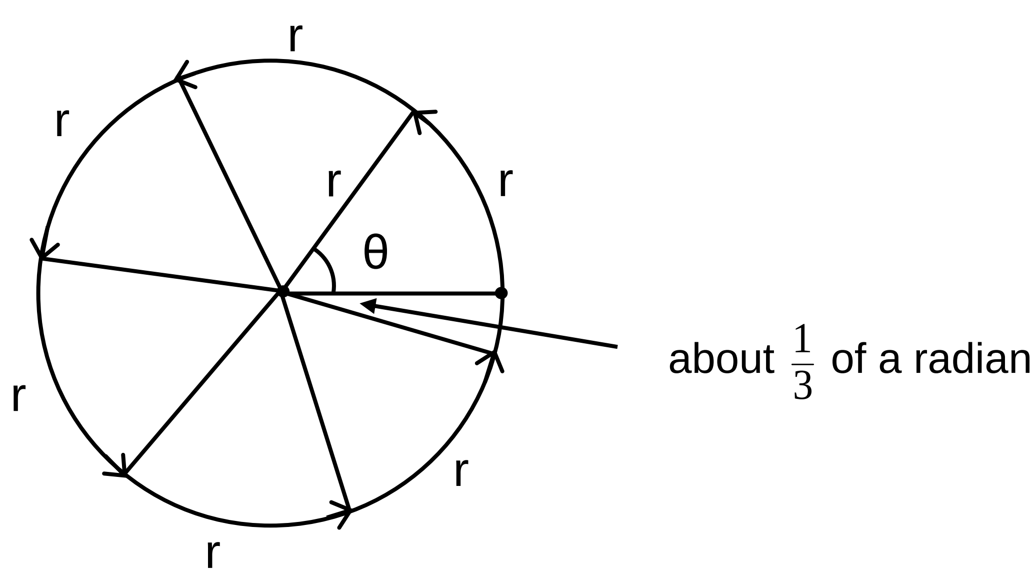


θ is 1 radian in size.

How many radians are in a circle?



θ is 1 radian

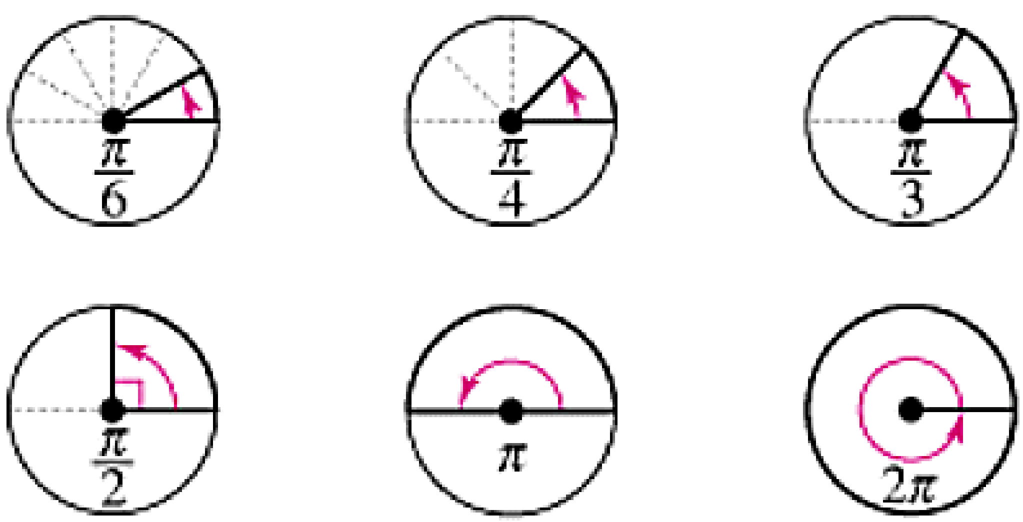


There are about $6\ \frac{1}{3}$ radians in a circle.

We know that the circumference of a circle is $2\pi r$.

This means that the circle itself contains an angle of rotation of 2π radians. Since 2π is approximately 6.28, this matches what we found above. There are a little more than 6 radians in a circle. (2π to be exact.)

Therefore: A <u>circle</u> contains 2π radians.
A <u>semi-circle</u> contains π radians of rotation.
A <u>quarter of a circle</u> (which is a right angle) contains $\pi/2$ radians of rotation.



<u>Definition</u>: A <u>degree</u> is a unit of angle measure that is equivalent to the rotation in 1/360th of a circle.

Because there are 360° in a circle, and we now know that there are also 2π radians in a circle, then $2\pi = 360°$.

$360° = 2\pi$ radians	2π radians = 360°
$180° = \pi$ radians	1π radians = 180°
$1° = \frac{\pi}{180}$ radians	1 radian = $\frac{180°}{\pi}$

To convert <u>radians to degrees</u>, multiply by $\frac{180°}{\pi}$.

To convert <u>degrees to radians</u>, multiply by $\frac{\pi}{180}$.

Example: Convert 120° to radians.

$$120° = 120\left(\frac{\pi}{180}\right) = \frac{120\pi}{180} = \frac{2\pi}{3}$$

Example: Convert -315° to radians.

$$-315° = -315\left(\frac{\pi}{180}\right) = \frac{-315\pi}{180} = \frac{-7\pi}{4}$$

Example: Convert $\frac{5\pi}{6}$ to degrees.

$$\frac{5\pi}{6} = \frac{5\pi}{6}\left(\frac{180}{\pi}\right) = \frac{900}{6} = 150°$$

Example: Convert 7 to degrees.

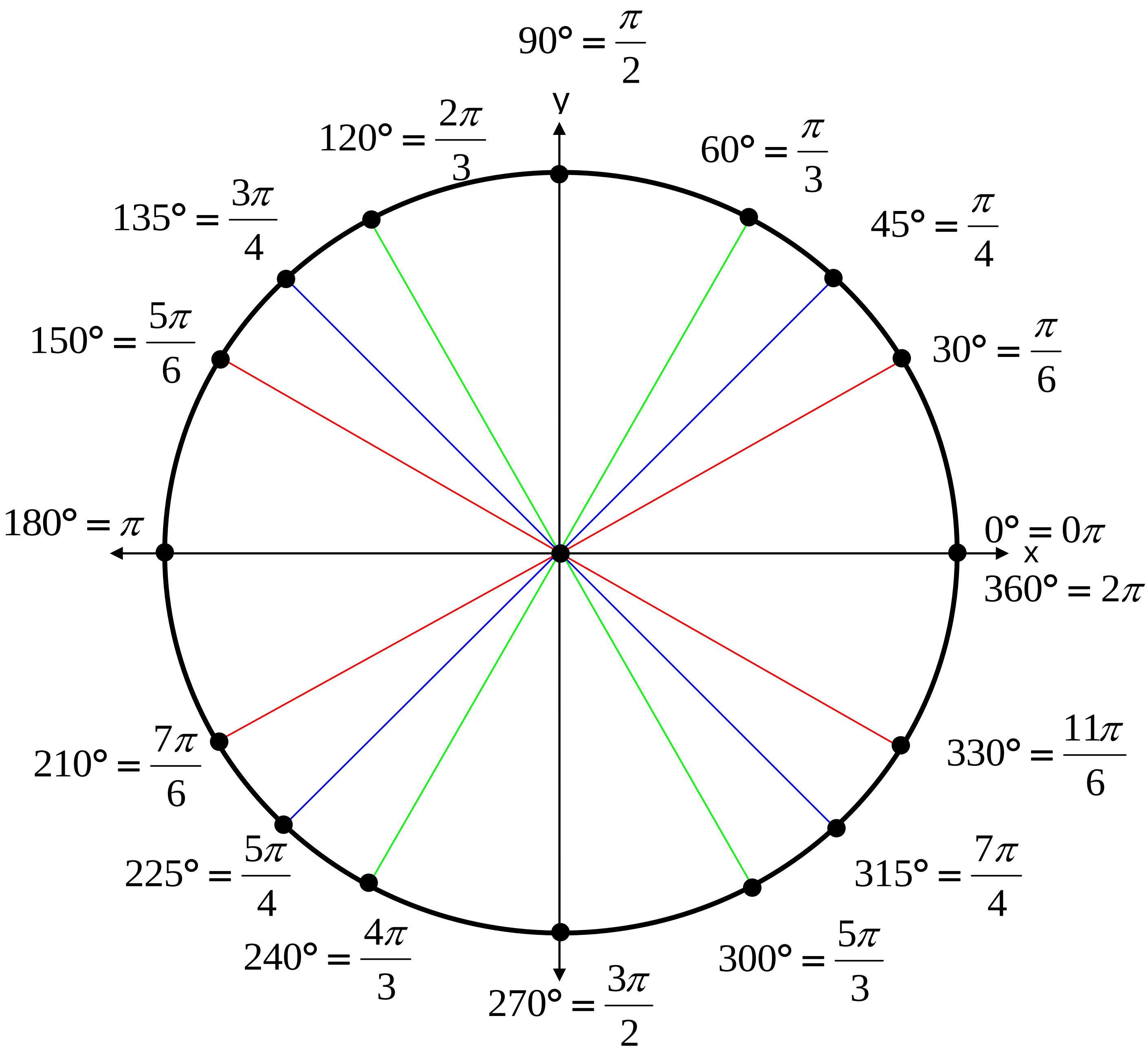
$$7 = 7\left(\frac{180}{\pi}\right) = \frac{1260}{\pi} = 401.07°$$

This makes sense, because 7 radians would be a little more than a complete circle, and 401.07° is a little more that 360°

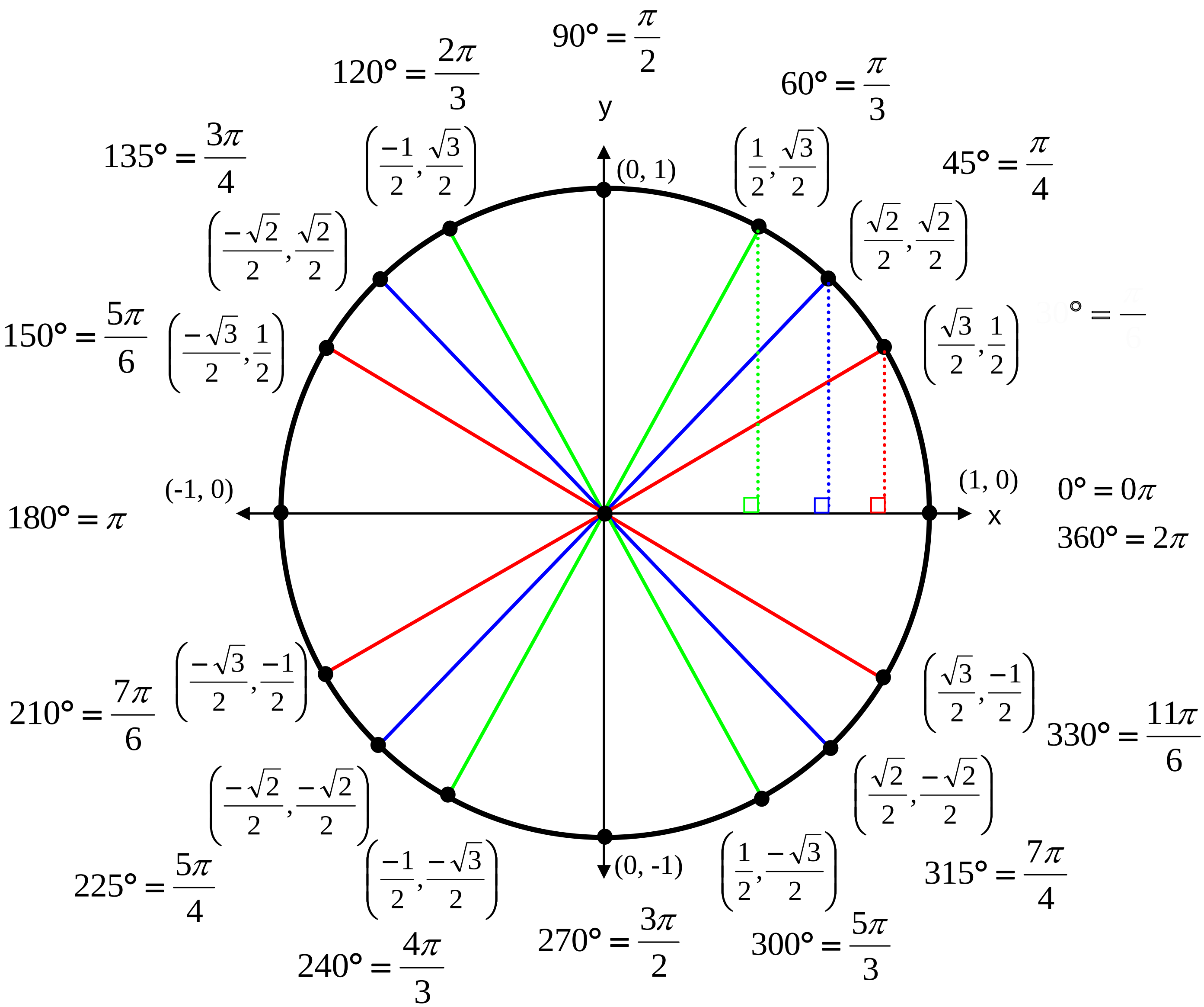
<u>*Notice</u>: If there is no unit specified, it is assumed to be radians.

*It is important to be familiar with the most common angles in radians and degrees.

Degree and Radian Equivalent measures



The Unit Circle



The cosine of angle θ is the *x*-coordinate.
The sine of angle θ is the *y*-coordinate.

$(x,y) = (\cos\theta, \sin\theta)$

Example: Using the unit circle, find the following.

a) $\sin\frac{3\pi}{4}$ answer : $\frac{\sqrt{2}}{2}$

b) $\cos\frac{-\pi}{2}$ answer : 0

c) $\tan 300°$ answer : $\frac{-\sqrt{3}}{2}\cdot\frac{2}{1} = -\sqrt{3}$

d) $\csc\pi$ answer : $\frac{1}{0} = \mathit{undefined}$