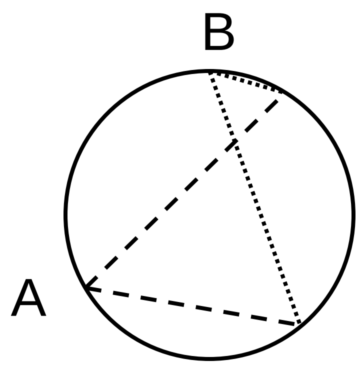
section 9.6



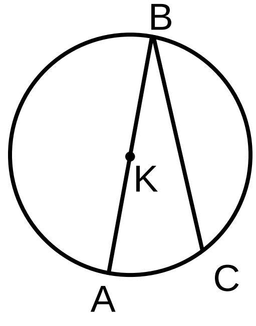
∠A and ∠B are inscribed. How are their angle measures related?

Answer: They are equal.

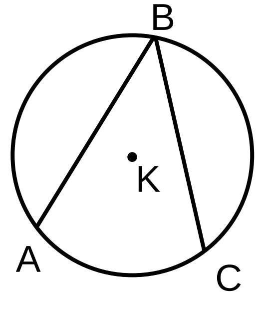
<u>Theorem 9.13</u>: The measure of an inscribed angle is equal to one-half the measure of its intercepted arc.

****To prove this theorem we must consider 3 possible cases. Consider circle K:**

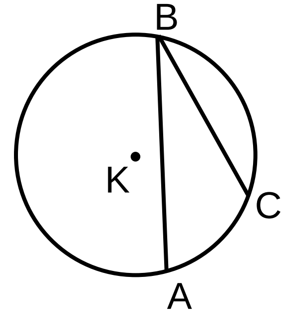
<u>Case 1</u>**:**
K lies on ∠ABC



<u>Case 2</u>**:**
K lies in the interior of ∠ABC



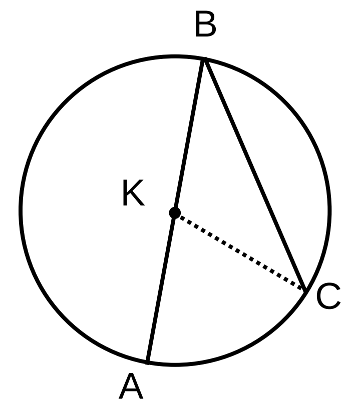
<u>Case 3</u>**:**
K lies in the exterior of ∠ABC



Proof of Case 1:

***Given*:** Circle K with inscribed $\angle ABC$ that intercepts $\overset{\frown}{AC}$

***Prove*:** $m\angle KBC = \frac{1}{2}\, m\overset{\frown}{AC}$



Statement	Reason
1. Circle K with inscribed $\angle ABC$ that intercepts $\overset{\frown}{AC}$	1. Given
2. K lies on $\angle ABC$	2. Given for Case 1
3. Draw $\overleftrightarrow{KC}$	3. Auxiliary line
4. $\overline{KB} \cong \overline{KC}$	4. Radii of circle are $\cong$
5. ΔKBC is isosceles Δ	5. Def. of isosceles Δ
6. $\angle KBC \cong \angle BCK$	6. Isosceles Δ Thm.
7. $m\angle KBC = m\angle BCK$	7. Def. of $\cong$ angles
8. $m\angle KBC + m\angle BCK = m\angle CKA$	8. Exterior $\angle$ Thm.
9. $m\angle CKA = m\overset{\frown}{AC}$	9. Def. of arc measure
10. $m\angle KBC + m\angle KBC = m\overset{\frown}{AC}$	10. Substitution (steps 7 and 9 into 8)
11. $2m\angle KBC = m\overset{\frown}{AC}$	11. Distributive
12. $m\angle KBC = \frac{1}{2}\, m\overset{\frown}{AC}$	12. Mult. prop. of equality

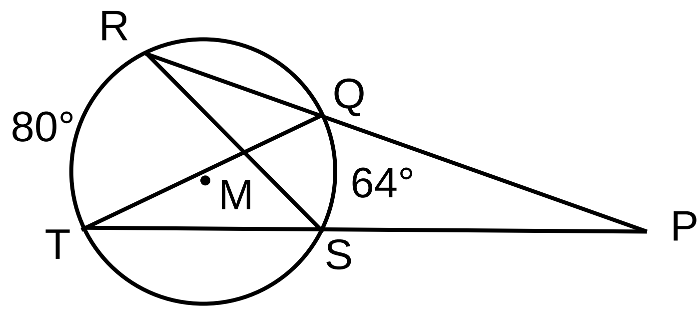
*Corollaries are theorems that follow immediately from some other theorem. Corollaries of Theorem 9.13:

Theorem 9.14: If two inscribed angles intercept congruent arcs, then the angles are congruent.

Theorem 9.15: An angle inscribed in a semicircle is a right angle.

Theorem 9.16: The opposite angles of an inscribed quadrilateral are supplementary.

Sample Problem: Given circle M, $m\overset{\frown}{RT} = 80°$,
$m\overset{\frown}{SQ} = 64°$



1. Find m∠QTS.

   $m\angle QTS = \frac{1}{2} m\overset{\frown}{QS} = \frac{1}{2}(64) = 32°$

2. Find m∠TQR.

   $m\angle TQR = \frac{1}{2} m\overset{\frown}{RT} = \frac{1}{2}(80) = 40°$

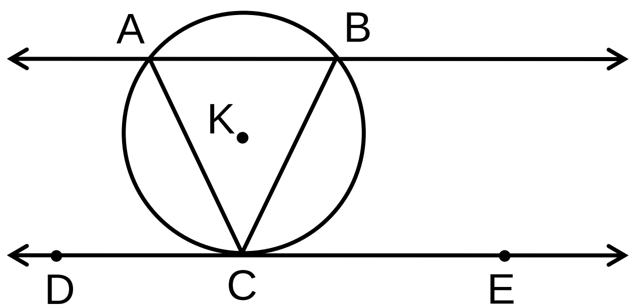
3. Find m∠TQP.

   $m\angle TQP = 180 – 40 = 140°$
   (∠TQP & ∠RQT are supp.)

4. Find m∠TPR.

   $32 + 140 + m\angle TPR = 180$, so $m\angle TPR = 8°$

Sample Problem: Given circle K, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$ (tangent), $\overline{AC} \cong \overline{BC}$, $m\angle BAC = 56°$



1. Find $m\overset{\frown}{AC}$.

   $m\angle B = 56°$ (isos. Δ) so $m\overset{\frown}{AC} = 2(56) = 112°$

2. Find $m\overset{\frown}{BC}$.

   Since $\overline{AC} \cong \overline{BC}$, their subtended arcs are $\cong$ and $m\overset{\frown}{BC} = 112°$

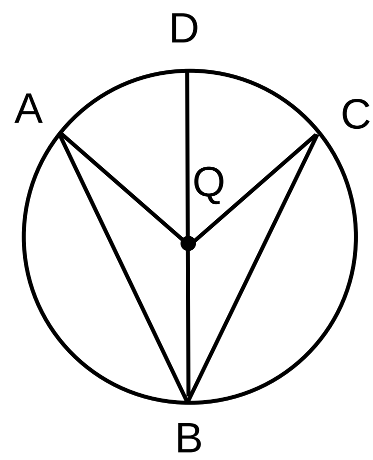
3. Find $m\angle ACB$.

   Look at ΔABC. $m\angle ACB = 180° - 56° - 56° = 68°$

4. Find $m\overset{\frown}{AB}$.

   $m\overset{\frown}{AB} = 2m\angle ACB = 2(68) = 136°$

Sample Problem:



Given**:** In circle Q,
$\overline{DB}$ is diameter,
$\overset{\frown}{AD} \cong \overset{\frown}{DC}$

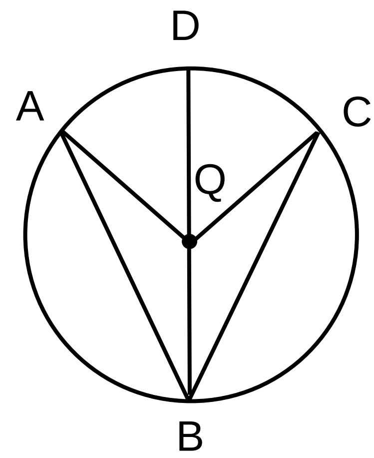
Prove**:** $\Delta ABQ \cong \Delta CBQ$

Statement	Reason
1. In circle Q, $\overline{DB}$ is diameter, $\overset{\frown}{AD} \cong \overset{\frown}{DC}$	1. Given
2.	2.
3.	3.
4.	4.
5.	5.
6.	6.
7.	7.
8.	8.

<u>Solution</u>:

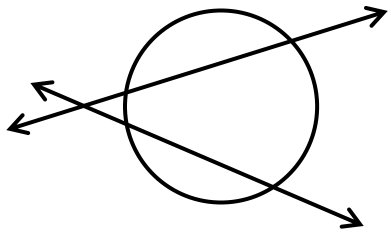
Given: In circle Q,
$\overline{DB}$ is diameter,
$\overset{\frown}{AD} \cong \overset{\frown}{DC}$

Prove: $\Delta ABQ \cong \Delta CBQ$

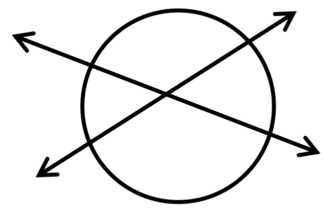


Statement	Reason
1. In circle Q, $\overline{DB}$ is diameter, $\overset{\frown}{AD} \cong \overset{\frown}{DC}$	1. Given
2. $\angle AQD \cong \angle CQD$	2. 2 minor arcs are ≅ iff central ∠'s are ≅
3. $\angle AQD$ & $\angle AQB$ are supp. $\angle CQD$ & $\angle CQB$ are supp.	3. Linear pairs are supplementary
4. $\angle AQB \cong \angle CQB$	4. Supplements of ≅ ∠'s are ≅
5. $\overline{AQ} \cong \overline{QC}$	5. All radii of circle ≅
6. $\overline{QB} \cong \overline{QB}$	6. Reflexive
7. $\Delta AQB \cong \Delta CQB$	7. SAS

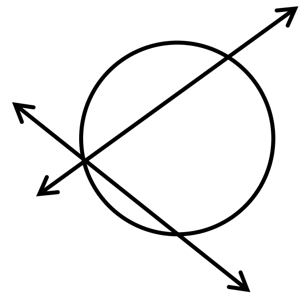
Look at how 2 intersecting lines can intersect a circle:



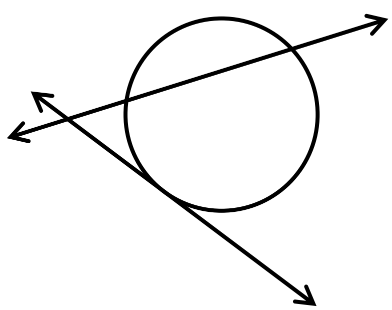
2 secants intersect at an exterior point



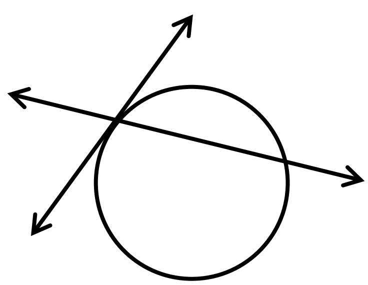
2 secants intersect at an interior point



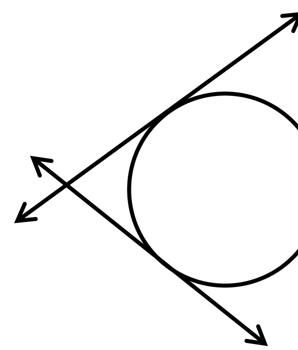
2 secants intersect at a point on the circle



secant and tangent intersect at an exterior point



secant and tangent intersect at a point on the circle



2 tangents intersect at and exterior point

How do we measure the angles made by these lines?

In General,

- If the point of intersection is exterior, then the measure of the angle is ½ of the difference of the intercepted arcs.

- If the point of intersection is on the circle, then the measure of the angle is ½ of the measure of the intercepted arc.

- If the point of intersection is interior, then the measure of the angle is ½ of the sum of the intercepted arcs.

These are included in the following theorems:

Theorem 9.17a: The measure of an angle formed by two secants that intersect in the exterior of a circle is one-half the difference of the measures of the intercepted arcs.

Theorem 9.17b: The measure of an angle formed by a secant and tangent that intersect in the exterior of a circle is one-half the difference of the measures of the intercepted arcs.

Theorem 9.17b: The measure of an angle formed by two tangents is one-half the difference of the measures of the intercepted arcs.

*We put these together into one theorem:

Theorem 9.17: If 2 lines intersect a circle and intersect each other in the exterior of the circle, then the angle formed measures one-half the difference of the measures of the intercepted arcs.

*For the other types of line/circle intersections:

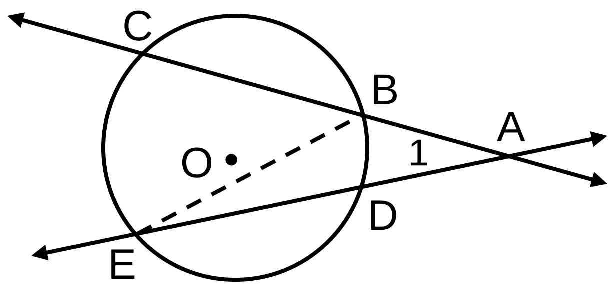
Theorem 9.18: The measure of an angle formed by two secants that intersect in the interior of a circle is one-half the sum of the measures of the intercepted arcs.

Theorem 9.19: The measure of an angle formed by a secant and tangent that intersect at the point of tangency is one-half the measure of the intercepted arc.

Proof of Theorem 9.17a:

Given: Circle O with secants $\overleftrightarrow{CA}$ and $\overleftrightarrow{EA}$

Prove: $m\angle 1 = \frac{1}{2}(m\overset{\frown}{CE} - m\overset{\frown}{BD})$

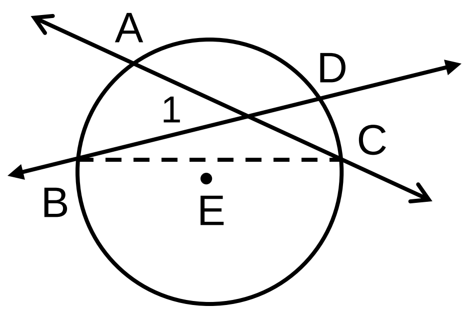


Statement	Reason
1. Circle O with secants $\overleftrightarrow{CA}$ and $\overleftrightarrow{EA}$	1. Given
2. Draw $\overline{BE}$	2. Auxiliary line
3. $m\angle CBE = m\angle BEA + m\angle 1$	3. Exterior $\angle$ Theorem
4. $m\angle 1 = m\angle CBE - m\angle BEA$	4. Add. Prop. of Eq.
5. $m\angle CBE = \frac{1}{2} m\overset{\frown}{CE}$ $m\angle BEA = \frac{1}{2} m\overset{\frown}{BD}$	5. An inscribed $\angle$ measures ½ measure of intercepted arc
6. $m\angle 1 = \frac{1}{2} m\overset{\frown}{CE} - \frac{1}{2} m\overset{\frown}{BD}$	6. Substitution (5 into 4)
7. $m\angle 1 = \frac{1}{2}(m\overset{\frown}{CE} - m\overset{\frown}{BD})$	7. Distributive

Proof of Theorem 9.18

***Given*:** $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are secants that intersect inside circle E and form $\angle 1$

***Prove*:** $m\angle 1 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{DC})$



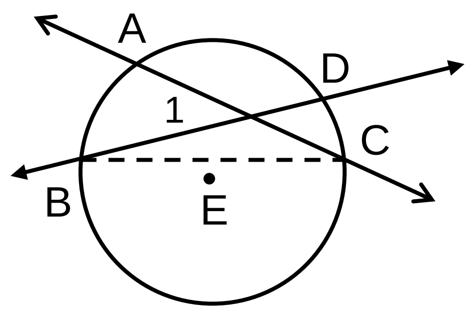
Statement	**Reason**
1.	1.
2.	2.
3.	3.
4.	4.
5.	5.
6.	6.

Solution:

Proof of Theorem 9.18

***Given*:** $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are secants that intersect inside circle E and form ∠1

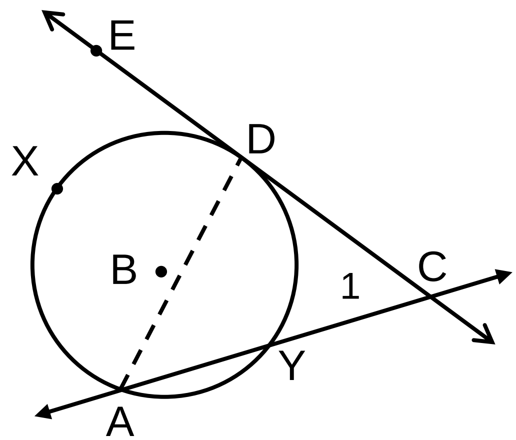
***Prove*:** m∠1 = ½ (m$\overset{\frown}{AB}$ + m$\overset{\frown}{DC}$)



Statement	Reason
1. $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are secants that intersect inside circle E and form ∠1	1. Given
2. Draw $\overline{BC}$	2. Auxiliary line
3. m∠1= m∠ACB + m∠DBC	3. Exterior angle Thm.
4. m∠ACB = ½ m$\overset{\frown}{AB}$ m∠DBC = ½ m$\overset{\frown}{DC}$	4. Inscribed ∠'s measure ½ the intercepted arc
5. m∠1= ½ m$\overset{\frown}{AB}$+ ½ m$\overset{\frown}{DC}$	5. Substitution (4 into 3)
6. m∠1= ½ (m$\overset{\frown}{AB}$ + m$\overset{\frown}{DC}$)	6. Distributive

Proof of Theorem 9.17b

***Given*:** $\overleftrightarrow{DC}$ is a tangent and $\overleftrightarrow{AC}$ is a secant to circle B
***Prove*:** $m\angle 1 = \frac{1}{2}(m\overparen{AXD} - m\overparen{DY})$

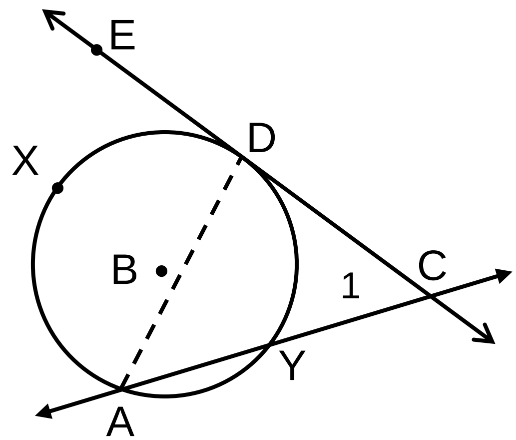


Statement	Reason
1. $\overleftrightarrow{DC}$ is a tangent and $\overleftrightarrow{AC}$ is a secant to circle B	1. Given
2.	2.
3.	3.
4.	4.
5.	5.
6.	6.
7.	7.
8.	8.

Solution:

Proof of Theorem 9.17b

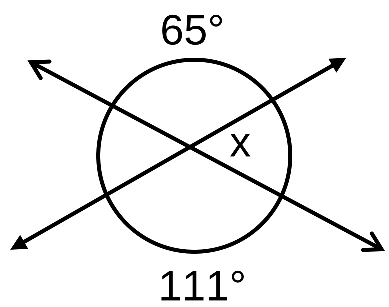
Given**:** $\overleftrightarrow{DC}$ is a tangent and $\overleftrightarrow{AC}$ is a secant to circle B
Prove**:** $m\angle 1 = \frac{1}{2}(m\overset{\frown}{AXD} - m\overset{\frown}{DY})$



Statement	Reason
1. $\overleftrightarrow{DC}$ is a tangent and $\overleftrightarrow{AC}$ is a secant to circle B	1. Given
2. Draw $\overleftrightarrow{AD}$	2. Auxiliary line
3. $m\angle ADE = m\angle CAD + m\angle 1$	3. Ext. Angle Thm.
4. $m\angle ADE - m\angle CAD = m\angle 1$	4. Add. Prop. of Eq.
5. $m\angle ADE = \frac{1}{2} m\overset{\frown}{AXD}$	5. $\angle$ meas. for secant and tangent at pt. of tan.
6. $m\angle CAD = \frac{1}{2} m\overset{\frown}{DY}$	6. Inscribed $\angle$ measures ½ intercepted arc
7. $\frac{1}{2} m\overset{\frown}{AXD} - \frac{1}{2} m\overset{\frown}{DY} = m\angle 1$	7. Substitution (5,6 into 4)
8. $m\angle 1 = \frac{1}{2}(m\overset{\frown}{AXD} - m\overset{\frown}{DY})$	8. Distributive

<u>Sample Problems</u>: Find x.

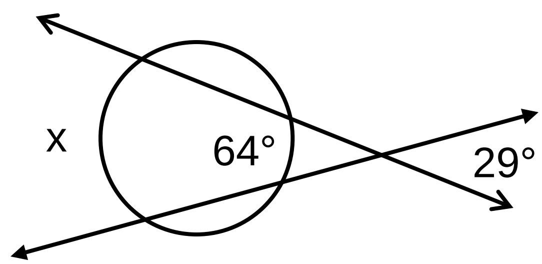
1.



sum of other 2 arcs = 360-65-111 = 184°

m∠X = ½ sum = ½ (184) = 92°

2.



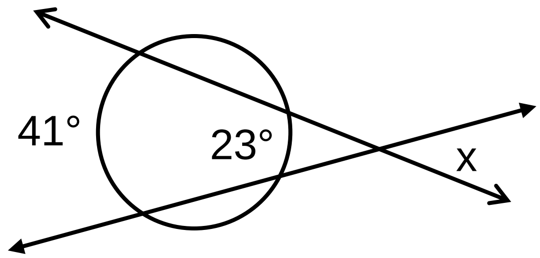
29 = ½ (x-64)

29 = ½ x – 32

½ x = 61

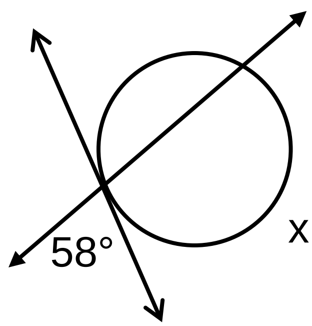
x = 122°

3.



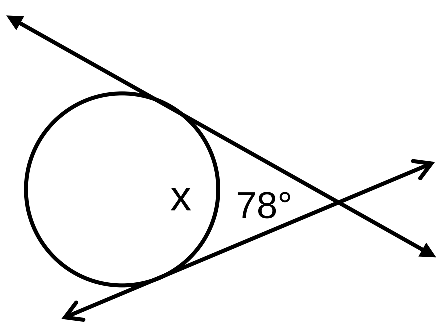
x = ½ (41-23) = ½ (18) = 9°

4.



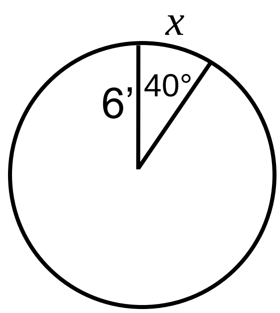
180-58 = 122
122 = ½ x, so x = 244°

5.



78 = ½ (360-x-x)
78 = 180 – x
x = 102°

Finding the lengths of arcs:



$x = \frac{40°}{360°}$ of circle

$x = \frac{1}{9}$ of circle

x = $\frac{1}{9}$ of 2πr (circumference)

x = $\frac{1}{9}$ of 12π

x = $\frac{12\pi}{9} = \frac{4\pi}{3}$ units

Theorem 9.20: If the degree measure of an arc is θ and the circumference of the circle is c, then the length of the arc is ℓ, given by

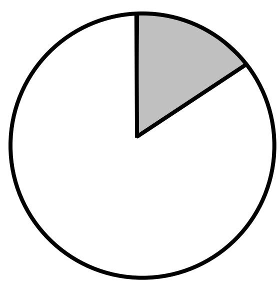
$$\frac{\ell}{c} = \frac{\theta}{360°} \quad \text{or} \quad \frac{\ell}{c} = \frac{\pi r\theta}{180°}$$

**Another form of this would be $\ell = \frac{\pi r\theta}{180°}$

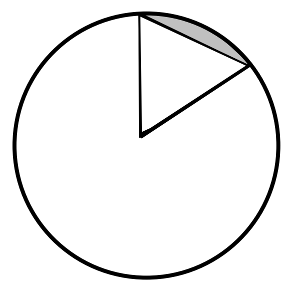
<u>Definitions</u>:

A <u>sector of a circle</u> is the region bounded by 2 radii and the intercepted arc.

A <u>segment of a circle</u> is the region bounded by a chord and is intercepted arc.

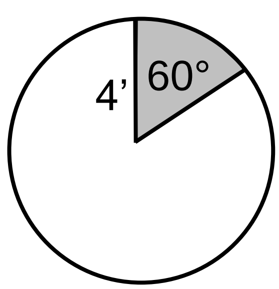


sector



segment

Finding the area of a sector:



Area = $\frac{\mathbf{60°}}{\mathbf{360°}}$ of circle

Area = $\frac{\mathbf{1}}{\mathbf{6}}$ of circle

Area = $\frac{\mathbf{1}}{\mathbf{6}}$ of πr^2

Area = $\frac{\mathbf{1}}{\mathbf{6}}$ of 16π

Area = $\frac{\mathbf{8\pi}}{\mathbf{3}}$ sq. inches

Theorem 9.21: The area of a sector is given by the proportion:

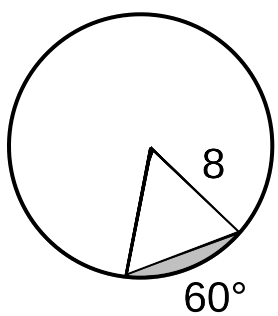
$$\frac{A}{A_c} = \frac{\theta}{360°} \qquad \textbf{or} \qquad A = A_c \frac{\theta}{360°}$$

where A is the area of the sector, A_c is the area of the circle, and θ is the arc measure in degrees.

**Another form of this would be $A = \frac{\pi r^2 \theta}{360°}$

Finding the area of a segment:

area of segment = area of sector – area of Δ



area of sector $= \frac{\pi r^2 \theta}{360} = \frac{32\pi}{3}$

area of equilateral $\Delta = \frac{s^2\sqrt{3}}{4} = 16\sqrt{3}$

area of segment = $\frac{32\pi}{3} - 16\sqrt{3} \approx 5.8$ square units

Sample Problems:

1. Find the area of the sector of a circle formed by a 20° central angle if the circle has diameter 12 units.

A = $\frac{\pi r^2\theta}{360} = \frac{36\pi(20)}{360} = 2\pi$ sq. units

2. Find the length of the arc of a circle with radius 10 if the arc measures 80°.

$\ell = \frac{\pi r^2\theta}{360} = \frac{40\pi}{9}$ units

3. Find the radius of a circle containing a sector with an area of 50 sq. meters and an arc of 24°.

$$A = \frac{\pi r^2\theta}{360}$$

$$50 = \frac{\pi r^2(24)}{360}$$

$$r^2 = \frac{750}{\pi}$$

$r = \sqrt{\frac{750}{\pi}} \approx 15.45$ sq. meters

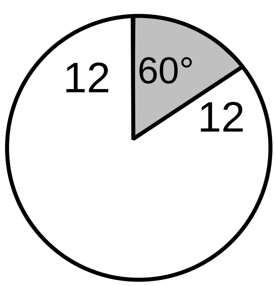
4. Find the degree measure of the arc for the sector with an area of 96 sq. inches in a circle with radius 8 inches.

$$A = \frac{\pi r^2 \theta}{360}$$

$$96 = \frac{\pi (64)\theta}{360}$$

$$\theta = \frac{540}{\pi} \approx 171.89°$$

Finding the perimeter of a sector:



Perimeter = 12 + 12 + **l**

P = 24 + $\frac{60}{360}$(2π)(12)

P = 24 + $\frac{1}{6}$(24π)

P = 24 + 4π

P ≈ 36.6 units

Sample Problem: Find the perimeter of a sector with an arc of 20° in a circle with radius 6.

$$\ell = \frac{20}{360} 2\pi(6) = \frac{2\pi}{3} \text{ units}$$
$$P = 6 + 6 + \frac{2\pi}{3}$$
$$P \approx 14.09 \text{ units}$$