Functions and Their Graphs

Definitions:

- A <u>relation</u> is a set of ordered pairs
- The <u>domain</u> of a relation is the set of all x-coordinates.
- The <u>range</u> of a relation is the set of all y-coordinates.
- A <u>function</u> is a relation in which each domain element corresponds to one and only one range element. (i.e. each x-coordinate can have only one y-coordinate.)

Example: S = {(1, 3),(-5, 4), (7, 3)}

a) What is the domain of S?

{-5, 1, 7}

b) What is the range of S?

{3, 4}

c) Is the relation S a function?

Yes

Definitions:

In a relation, the variable representing the domain elements is called the independent variable and the variable representing the range elements is called the dependent variable.

Function Tests

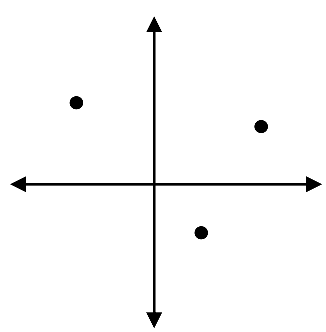
A relation is usually represented in one of the following ways:

1. A list of the ordered pairs in set notation
2. A graph of the ordered pairs
3. An equation that represents the ordered pairs

Examples

Ordered pairs: S={(2,3), (4, -3), (7, 3), (9,3)}

Graph:



Equation: y = 2x – 5

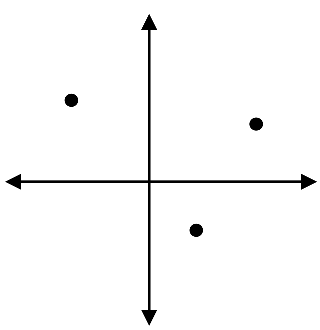
There is a test for each representation.

Function Tests

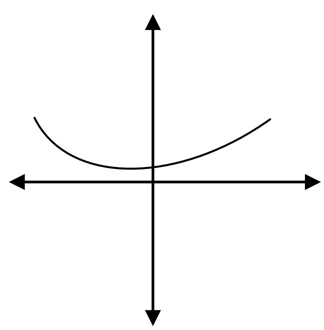
1. A list of the ordered pairs – check for unique x's
2. A graph of the ordered pairs – Vertical Line Test
3. An equation – Function Machine

Vertical Line Test – If every vertical line meets the graph of a relation in at most one point, then the relation is a function.

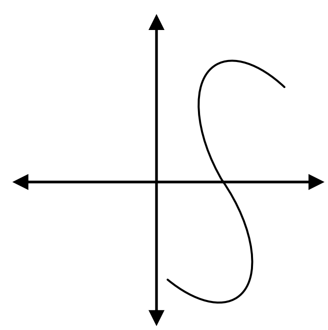
Example: Which of the following represents a function?



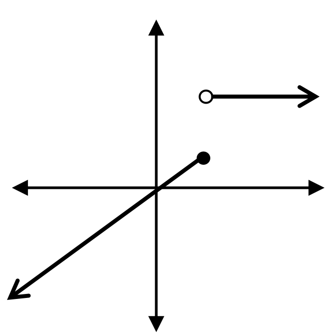
yes



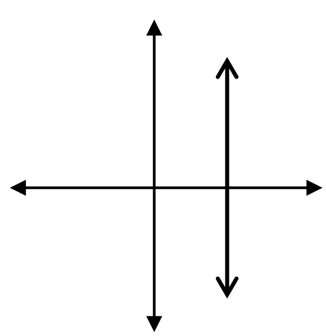
yes



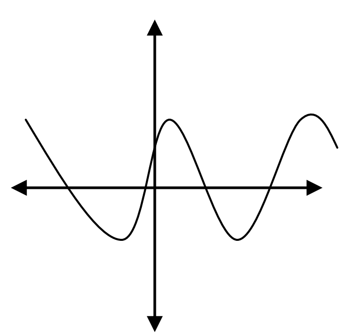
no



yes



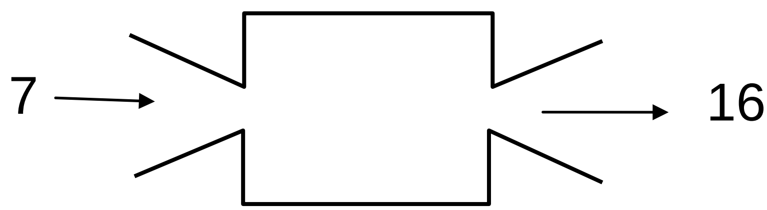
no



yes

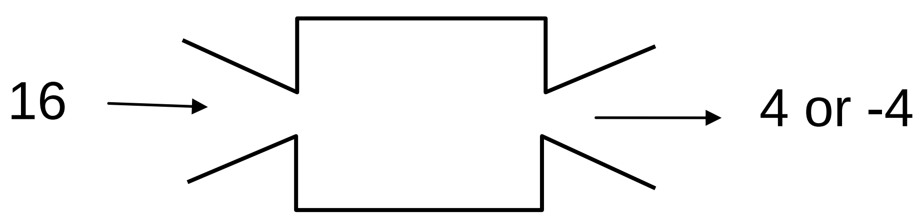
Function Machine – If you put a value for x into the machine, and know exactly which single number will come out, then the equation represents a function.

Example: Does $y = 3x - 5$ represent a function?



When we put in a number, we know exactly what comes out, so it is a function.

Example: Does $x = y^2$ represent a function?



This is not a function.

Note: In general, if you can solve an equation uniquely for y, it is a function.

In the above example, if we solve $x = y^2$ for y, we get

$$y = \pm\sqrt{x}$$

This is not solved uniquely because of the $\pm$.
Thus, it is not a function.

Function Notation

It is often necessary to give functions names so that they can be referenced easily. For relations we use a capital letter, but for functions we use a lower-cased letter.

Example: The equation $y = 3x - 5$ can be given the name f.

We write: $f(x) = 3x - 5$

This is read "f at x" or "f of x."

We say that for the function f, when we put in x, we get out the value of $3x - 5$.

2 Important notes about function notation:

1. f is the name of the function, not a variable.
2. $f(x)$ is just another way of saying y.

Example: Write $y = x^2 - 4x$ using function notation.

Solution: $f(x) = x^2 - 4x$

Example: Given $f(x) = x^2 - 4x$, find the following.

a) $f(2)$

$$f(x) = x^2 - 4x$$
$$f(2) = 2^2 - 4(2)$$
$$f(2) = 4 - 8$$
$$f(2) = -4$$

This means that when x = 2, then y = -4.

b) $f(k)$

$$f(x) = x^2 - 4x$$
$$f(k) = k^2 - 4k$$

It cannot be written any simpler.

c) $f(x+h)$

$$f(x) = x^2 - 4x$$
$$f(x+h) = (x+h)^2 - 4(x+h)$$
$$f(x+h) = x^2 + 2hx + h^2 - 4x - 4h$$

The following example uses a piece-wise function.

Example: Given $g(x) = \begin{cases} x+1, & \text{if } x > 0 \\ -x, & \text{if } x \le 0 \end{cases}$

a) Find $g(2)$

Since $2 > 0$, we use the top part of the function definition.

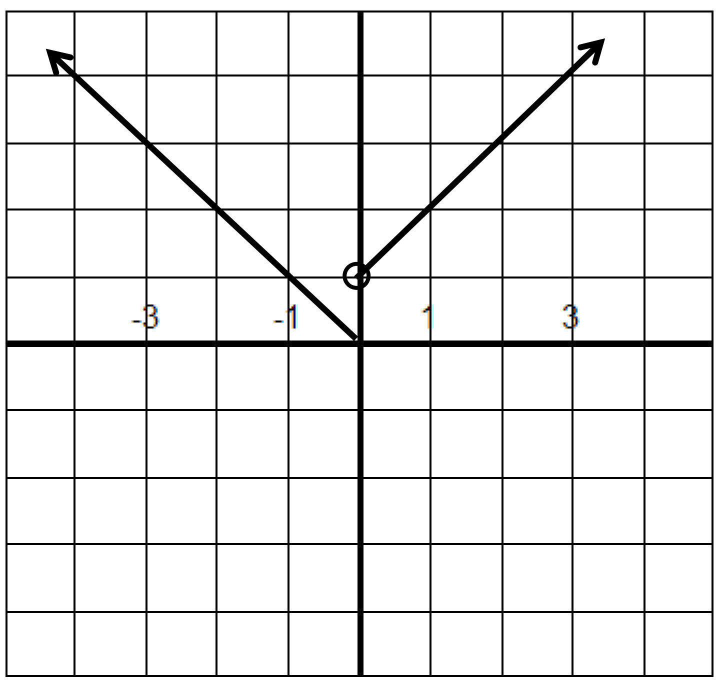
$$g(x) = x + 1$$
$$g(2) = 2 + 1$$
$$g(2) = 3$$

b) Find $g(-5)$

Since $-5 \le 0$, we use the top part of the function definition.

$$g(x) = -x$$
$$g(-5) = -(-5)$$
$$g(-5) = 5$$

The graph of this piecewise function is:



$$g(x) = \begin{cases} x+1, & \text{if } x > 0 \\ -x, & \text{if } x \le 0 \end{cases}$$

Graphing Piecewise Functions on the Graphing Calculator

For a piecewise function, enter each piece enclosed in parentheses, followed by the domain of the piece, also in parentheses. These 2 sets of parentheses are connected with other sets with a + sign.

[y=] ($piece_1$) ($domain_1$) + ($piece_2$) ($domain_2$) + …

For the example above, we have:

[y=] (x+1) (x>0) + ((-)x)(x ≤0)

(The calculator returns the value of 1 if the inequality is true and 0 if it's false.)

Notes:

- To access the inequality symbols, press [2nd] [TEST].
- It is permissible to use a double inequality such as (2 < x < 5).
- To have the calculator not draw a line connecting the 2 pieces, press [MODE] and then select Dot on the 5th line down. (Depending on the function, it may or may not get rid of the connecting line.)

The Domain of a Function

If the domain is not specifically given, we will use the implied domain.

Definition: The implied domain of a function is the set of all x such that the corresponding y is a real number.

For now, we will consider three situations:

1. Polynomials – the domain is the set of all real numbers, written $(-\infty,\infty)$ or $\mathbb{R}$. Fractions – the domain is all real numbers except those that make the denominator equal to zero.
2. Radicals – if the index is even, then the domain is all real numbers such that the radicand is nonnegative.

Example: Find the domain of the following functions.

a) $f(x) = x^3 + 3x + 1$

This is a polynomial, so the domain is $(-\infty,\infty)$.

b) $f(x)=\dfrac{2}{x+3}$

- This is a fraction. Since the denominator cannot equal zero, we must have $x+3\neq 0$.
- The domain is all real numbers except -3, which is written $(-\infty,-3)\cup(-3,\infty)$.

c) $f(x)=\sqrt[4]{5-x}$

- This is a radical with an even index. Since what is under the radical must be ≥ 0, we must have $5-x\geq 0$. Solving this inequality gives us $x\leq 5$.
- The domain is all real numbers less than or equal to 5, which is written $(-\infty,5]$

<u>Note</u>: In interval notation, the square bracket means the value is included. The parentheses means the value is not included. Infinity always gets parentheses.

Applications

Example: Given $f(x) = 3x + 2$, find $\dfrac{f(x+h) - f(x)}{h}$.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{[3(x+h)+2] - [3x+2]}{h} \\
&= \frac{3x + 3h + 2 - 3x - 2}{h} \\
&= \frac{3h}{h} \\
&= 3
\end{aligned}$$

Example: Given $f(x) = x^2 + x - 1$, find $\dfrac{f(x+h) - f(x)}{h}$.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 + (x+h) - 1] - [x^2 + x - 1]}{h} \\
&= \frac{x^2 + 2hx + h^2 + x + h - 1 - x^2 - x + 1}{h} \\
&= \frac{2hx + h^2 + h}{h} \\
&= 2x + h + 1 \qquad (h \neq 0)
\end{aligned}$$