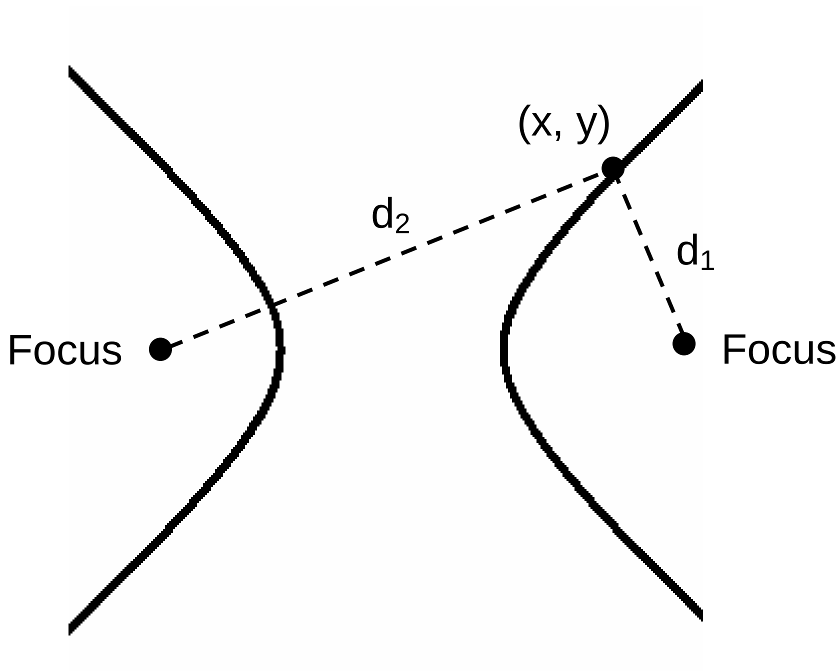
Hyperbolas

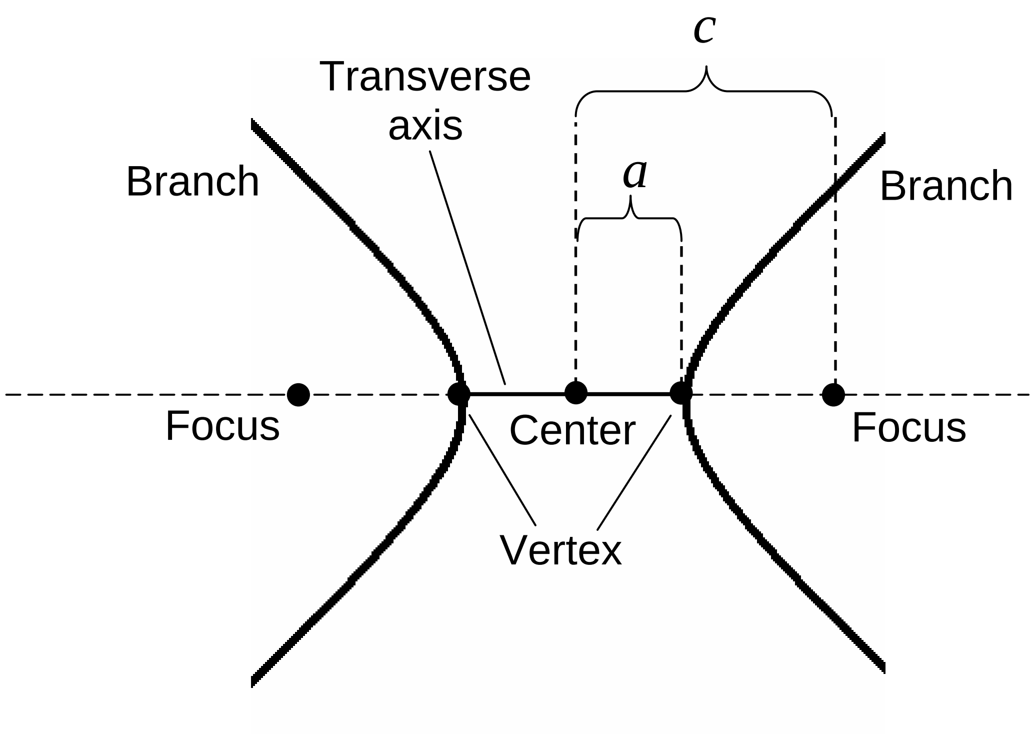
The third type of conic is called a hyperbola. For an ellipse, the sum of the distances from the foci and a point on the ellipse is a fixed number. For a hyperbola, the *difference* of the distances from the foci and a point on the hyperbola is a fixed number.

Definition of Hyperbola

A hyperbola is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points (foci) is a positive constant.



$d_2 - d_1$ is a constant.



- The midpoint between the foci is the <u>center</u>.

- The line segment joining the vertices is the <u>transverse axis.</u>

- The points at which the line through the foci meets the hyperbola are the <u>vertices</u>.

- The graph of the hyperbola has two disconnected <u>branches</u>.

- The vertices are a units from the center.

- The foci are c units from the center.

Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola centered at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$ Transverse axis is horizontal

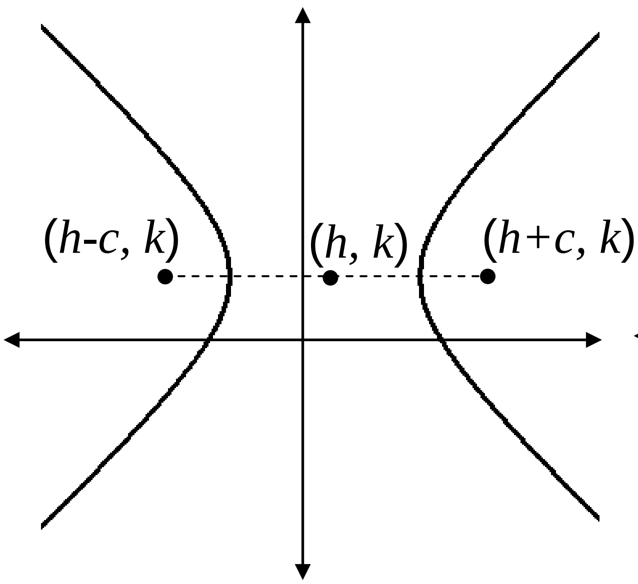
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$ Transverse axis is vertical

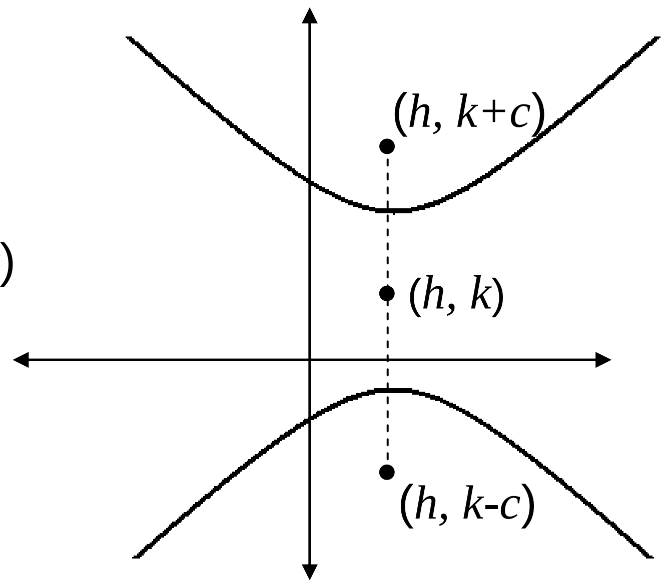
The vertices are a units from the center, and the foci are c units from the center, with $c^2 = a^2 + b^2$. If the center is at the origin (0, 0), the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ Transverse axis is horizontal

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$ Transverse axis is vertical

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$





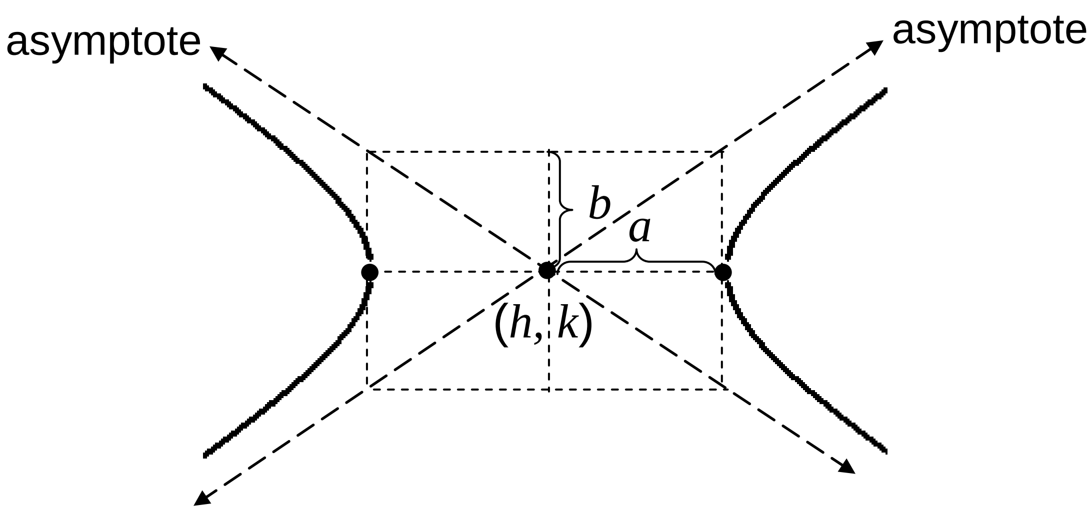
*In general, if the x-term is listed first, both branches of the hyperbola will cross the x-axis. If the y-term is listed first, both branches of the hyperbola will cross the y-axis.

What about *b*?

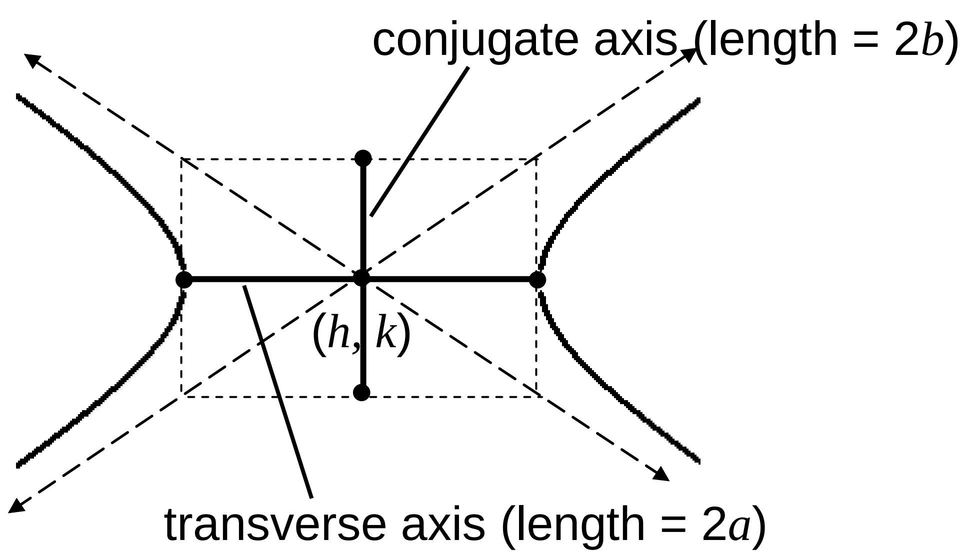
The value of b aids us in graphing the hyperbola by helping us find the asymptotes.

Asymptotes of a Hyperbola

Each hyperbola has two asymptotes that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle formed using the values of a and b and centered at (h, k).



There are 2 axes for a hyperbola.



Looking at the graph, and using slope = rise/run, we can see that the slope of the asymptotes is $\pm\frac{b}{a}$. This will always be the case when the transverse axis is horizontal. When the transverse axis is vertical the slope = $\pm\frac{a}{b}$.

Asymptotes of a Hyperbola

The equation for the asymptotes of a hyperbola are

$y = k \pm \frac{b}{a}(x-h)$ Transverse axis is horizontal

$y = k \pm \frac{a}{b}(x-h)$ Transverse axis is vertical

If the center of the hyperbola is at the origin, then the asymptotes are

$y = \pm\frac{b}{a}x$ Transverse axis is horizontal

$y = \pm\frac{a}{b}x$ Transverse axis is vertical

Example: Find the equation of the asymptotes of the hyperbola given by

$$\frac{(x-1)^2}{16}-\frac{(y+3)^2}{9}=1$$

Solution: The transverse axis is horizontal so the equation is of the form $y=k\pm\frac{b}{a}(x-h)$. Substituting the values from our equation gives us

$$y=-3\pm\frac{3}{4}(x-1)$$

Example: Find the equation of the asymptotes of the hyperbola given by

$$\frac{y^2}{25}-\frac{x^2}{4}=1$$

Solution: The center is at the origin and the transverse axis is vertical so the equation is of the form $y=\pm\frac{a}{b}x$. Substituting the values from our equation gives us $y=\pm\frac{5}{2}x$.

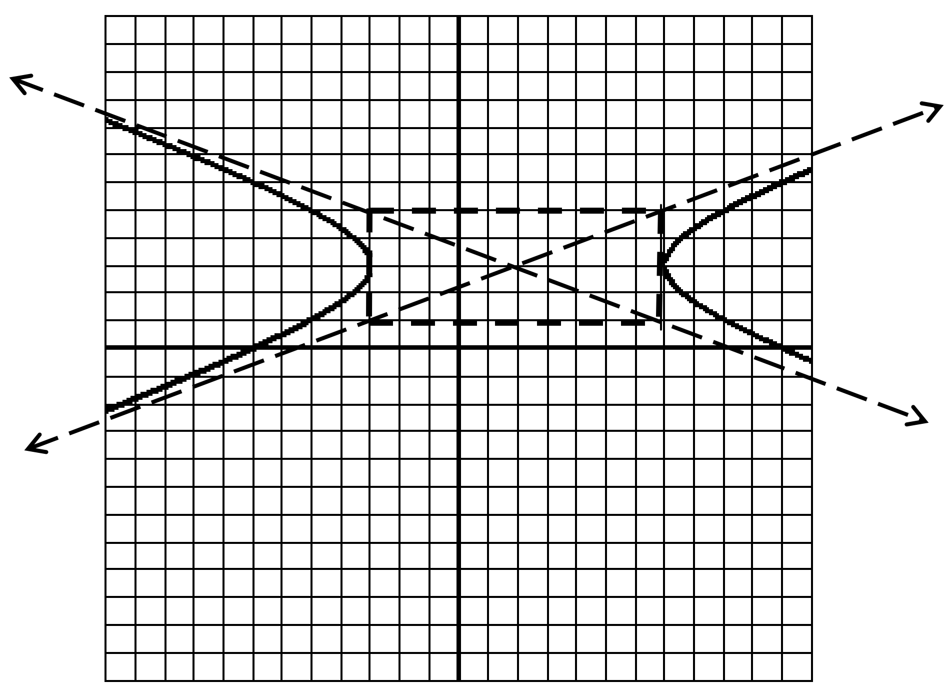
Example: Sketch the hyperbola given by the equation

$$\frac{(x-2)^2}{25}-\frac{(y-3)^2}{4}=1$$

Solution:

- The center is at (2, 3).
- The hyperbola opens right and left because the x-term is listed first.
- The transverse axis is 10 units, as $a = 5$
- The conjugate axis is 4 units, as $b = 2$.

Draw the box and sketch the asymptotes. Then sketch the hyperbola.



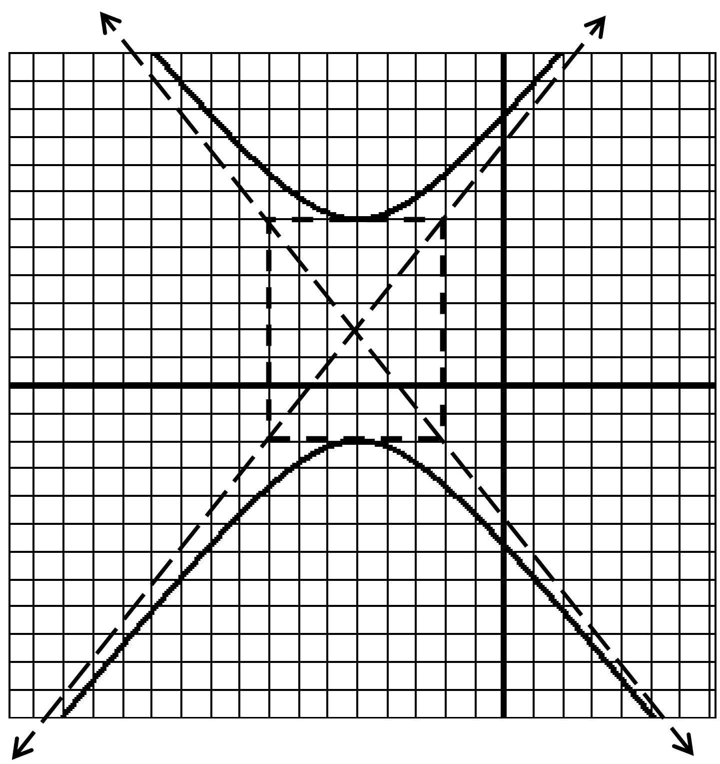
Example: Sketch the hyperbola given by the equation

$$\frac{(y-2)^2}{16}-\frac{(x+5)^2}{9}=1$$

Solution:

- The center is at (-5, 2).
- The hyperbola opens up and down because the y-term is listed first.
- The transverse axis is 8 units, as a = 4
- The conjugate axis is 6 units, as b = 3.

Draw the box and sketch the asymptotes. Then sketch the hyperbola.



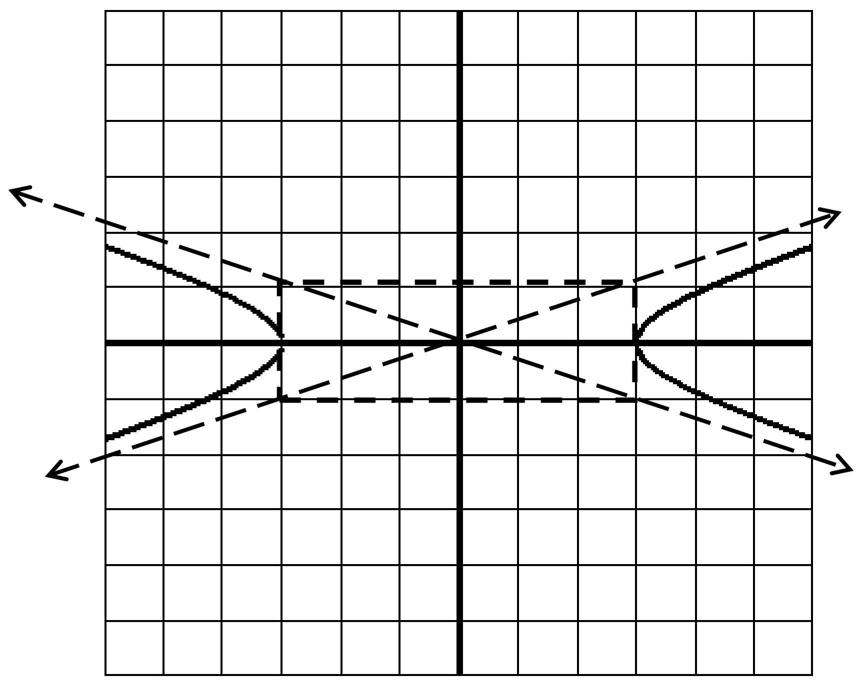
Example: Sketch the graph of the hyperbola given by

$$x^2 - 9y^2 = 9$$

Solution: Divide through by 9 to get 1 on the right.

$$x^2 - 9y^2 = 9$$

$$\frac{x^2}{9} - \frac{y^2}{1} = 1$$



Example: Find the standard form of the equation of the hyperbola with foci (-1, 2) and (5, 2) and vertices (0, 2) and (4, 2).

Solution: Sketch the given information. You will see that

- The center must be (2, 2).
- The transverse axis is horizontal.
- The standard equation must be of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

We know that a = 2 and c = 3, so we can find b.

$$c^2 = a^2 + b^2$$

$$3^2 = 2^2 + b^2$$

$$9 - 4 = b^2$$

$$b = \sqrt{5}$$

Therefore, our equation must be

$$\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1$$

Example: Sketch the graph of the hyperbola given by

$$4x^2 - 9y^2 - 24x - 72y - 72 = 0$$

<u>*Solution*</u>: Find the equation by completing the square.

*You must pull out -9 from the y-terms. Be careful.

$$4(x^2 - 6x + \_\_) - 9(y^2 + 8y + \_\_) = 72 + \_\_ + \_\_$$

*Be careful filling in your blanks on the right side.

$$4(x^2 - 6x + \underline{9}) - 9(y^2 + 8y + \underline{16}) = 72 + \underline{36} + \underline{-144}$$

$$4(x-3)^2 - 9(y+4)^2 = -36$$

*Divide through by -36 and write in correct form.

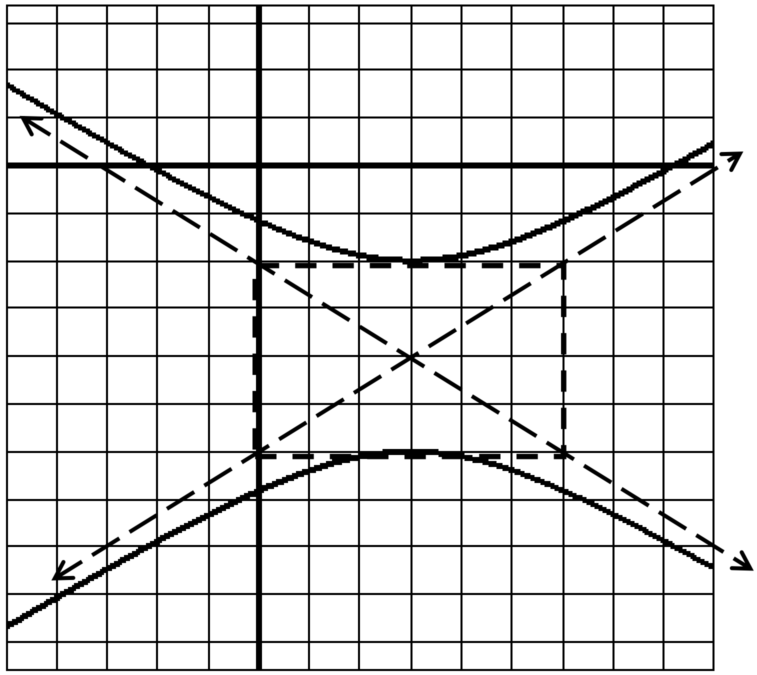
$$\frac{4(x-3)^2}{-36} - \frac{9(y+4)^2}{-36} = \frac{-36}{-36}$$

$$\frac{(x-3)^2}{-9} + \frac{(y+4)^2}{4} = 1$$

$$\frac{(y+4)^2}{4} - \frac{(x-3)^2}{9} = 1$$

Sketch the graph.

$$\frac{(y+4)^2}{4} - \frac{(x-3)^2}{9} = 1$$



Example: Find the equation of the hyperbola if the vertices are (0,3) and (0,-3) and the asymptotes are $y = 3x$ and $y = -3x$.

<u>*Solution*</u>: Graph the given information.

By the picture you can see that the hyperbola opens up and down, so the asymptotes are of the form $y = \pm \frac{a}{b} x$. Since we know $a = 3$ then we must have $b = 1$.

The equation is $\frac{y^2}{9} - \frac{x^2}{1} = 1.$

Example: Find the equation of the hyperbola if the foci are (0,10) and (0,-10) and the asymptotes are y = ¾ x and y = -¾ x.

Solution: By the picture you can see that the hyperbola opens up and down, so the asymptotes are of the form $y = \pm\frac{a}{b}x$. Since our equation is y = " ¾ x, we must have $\frac{a}{b} = \frac{3}{4}$. Solving gives us $a = \frac{3}{4}b$.

Solve $c^2 = a^2 + b^2$ since we have c = 10.

$$c^2 = a^2 + b^2$$

$$10^2 = \left(\frac{3}{4}b\right)^2 + b^2$$

$$100 = \frac{25}{16}b^2$$

$$b^2 = 64$$

$$b = 8$$

$$a = \frac{3}{4}b$$

$$a = \frac{3}{4}(8) = 6$$

The equation is

$$\frac{(y-0)^2}{6^2} - \frac{(x-0)^2}{8^2} = 1$$

$$\frac{y^2}{36} - \frac{x^2}{64} = 1$$

Example: Find the equation of the hyperbola with vertices (2,1) and (-2,1) and passes through the point ($\sqrt{5}$,0).

Solution: We can tell from the vertices that a = 2. We can also tell that the center is (0, 1). We know that the hyperbola opens left and right, so will be of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Put in all of the known information and solve for b.

$$\frac{(\sqrt{5}-0)^2}{2^2} - \frac{(0-1)^2}{b^2} = 1$$

$$\frac{5}{4} - \frac{1}{b^2} = 1$$

$$\frac{5}{4} - 1 = \frac{1}{b^2}$$

$$\frac{1}{b^2} = \frac{1}{4} \text{ so } b = 2$$

The equation is: $\frac{(x-0)^2}{4} - \frac{(y-1)^2}{4} = 1$

Graphing Conics Using a Graphing Calculator

Since the equation editor of your graphing calculator is in the form “y = “, you will have to solve all equations for y. This may mean taking the square root of both sides. If that is the case, you must graph the positive root in one equation and the negative root in the other.

Example: Graph $\frac{(y+4)^2}{4} - \frac{(x-3)^2}{9} = 1$ using a graphing calculator.

$$\frac{(y+4)^2}{4} = \frac{(x-3)^2}{9} + 1$$

$$(y+4)^2 = 4\left[\frac{(x-3)^2}{9} + 1\right]$$

$$\sqrt{(y+4)^2} = \pm\sqrt{4\left[\frac{(x-3)^2}{9} + 1\right]}$$

$$y+4 = \pm 2\sqrt{\left[\frac{(x-3)^2}{9} + 1\right]}$$

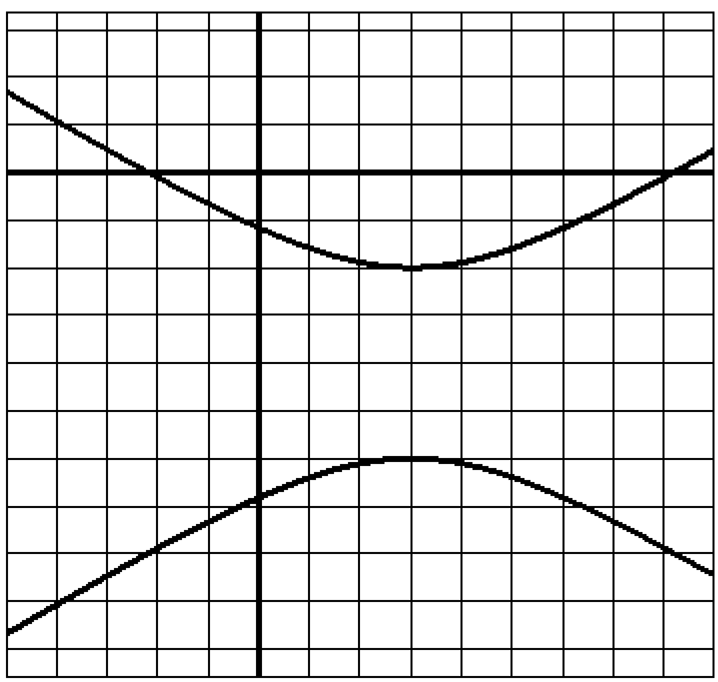
$$y = \pm 2\sqrt{\left[\frac{(x-3)^2}{9} + 1\right]} - 4$$

Graph this as the following 2 equations:

$$y_1 = 2\sqrt{\left[\frac{(x-3)^2}{9}+1\right]}-4 \quad \textbf{and} \quad y_2 = -2\sqrt{\left[\frac{(x-3)^2}{9}+1\right]}-4$$

Type the following in your equation editor:

$$y_1 = 2\sqrt{\ }(((X-3)^2/9)+1)-4$$
$$y_2 = -2\sqrt{\ }(((X-3)^2/9)+1)-4$$



You may have to adjust your viewing window.

Eccentricity

Definition of Eccentricity

The eccentricity e of a hyperbola is given by the ratio

$$e = \frac{c}{a}$$

*Note: Because c is always greater than a, the fraction $\frac{c}{a}$ will always be greater than 1.

- If the eccentricity is close to 1, the branches of the hyperbola are more pointed.
- If the eccentricity is great, the branches are flatter.

Example: Find the eccentricity of the hyperbola given by

$$\frac{(y+4)^2}{4} - \frac{(x-3)^2}{9} = 1$$

Solution: Solving for c gives us $e = \frac{c}{a} = \frac{\sqrt{13}}{2} \approx 1.8$

General Equations of Conics

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following:

1. A circle if $A = C$
2. A parabola if $AC = 0$ *(A=0 or C=0, but not both)*
3. An ellipse if $AC > 0$ *(A and C have like signs)*
4. A hyperbola if $AC < 0$ *(A and C have unlike signs)*

Example: Classify each of the following.

a) $4x^2 + 5y^2 - 9x + 8y = 0$

ellipse (AC = 20)

b) $2x^2 - 5x + 7y - 8 = 0$

parabola (AC = 0)

c) $7x^2 + 7y^2 - 9x + 8y - 16 = 0$

circle (A = C)

d) $4x^2 - 5y^2 - x + 8y + 1 = 0$

hyperbola (AC = -20)