Parametric Equations

Parametric equations are used when we want to consider a 3rd variable. Look at the example on page 742 of the text.

Definition of Plane Curve

If f and g are continuous functions of t on the interval I, the set of ordered pairs ($f(t), g(t)$) is a plane curve C. The equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are parametric equations for C, and t is the parameter.

Sketching a Plane Curve

One way of sketching plane curves is by point plotting. The advantage of this method is that we can see how the curve is traced in the order of increasing values of t, called the orientation of the curve.

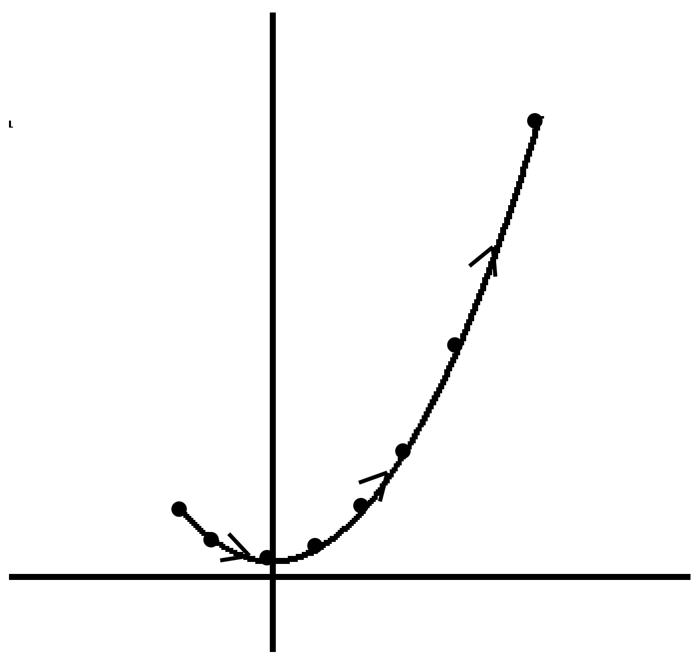
Example: Sketch the curve represented by the parametric equations

$$x = 2t \quad \text{and} \quad y = t^2 + 1 \quad \text{on the interval [-2, 5].}$$

Solution: Make a table.

t	x	y
-2	-4	5
-1	-2	2
0	0	1
1	2	2
2	4	5
3	6	10
4	8	17
5	10	26

Graph the ordered pairs (x, y).



Note that the arrows on the curve indicate its orientation as t increases from -2 to 5.

Graphing Parametric Equations using a Graphing Calculator

1. Press [MODE] and on the 4th line down, make sure that [Par] is selected.

2. Press [Y=]. You will see the parametric equation editor. Enter your equation for x and your equation for y. Press [GRAPH].

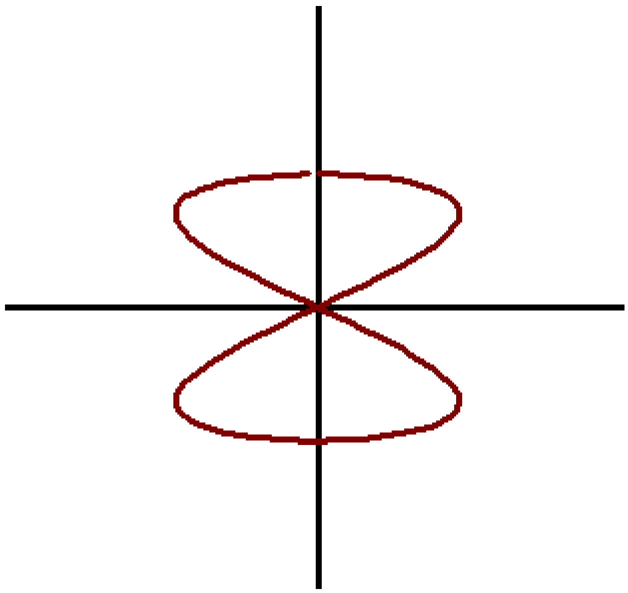
3. Adjust the window using the WINDOW button. Enter the range for t, if given. Adjust the viewing window by entering ranges for x and y, or using the ZOOM feature.

4. To find the orientation of the graph, use the TRACE feature. When you press [TRACE], you will be given the 2 parametric equations at the top of the window, and the values of t, x, and y at the bottom of the window. As you press the left and right arrows, you will be able to see the orientation as t increases.

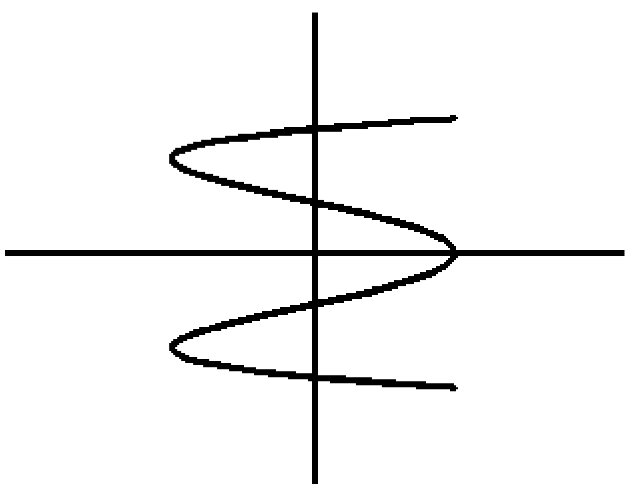
*Note: When graphing parametric equations involving trig functions, make sure the calculator is set for the correct angle measurement mode.

*Also Note: The value for Tstep in the WINDOW feature should generally be small, like 0.05. The larger the number, the less accurate the graph.

Example: On a graphing calculator, graph the parametric equations $x = 4\sin 2\theta$ and $y = 4\cos\theta$ for $0 \le \theta \le 2\pi$.



Example: On a graphing calculator, graph the parametric equations $x = \cos 4\theta$ and $y = \sin\theta$ for $0 \le \theta \le 2\pi$.



Trace to see the orientation.

Eliminating the Parameter

If you want to y as a function of x, you must eliminate the parameter. To do this, solve one of the 2 parametric equations for t and then substitute.

Example: Eliminate the parameter for the equations

$$x = 2t \quad \text{and} \quad y = t^2 + 1$$

Solution: Solve the first equation for t.

$$x = 2t$$

$$t = \frac{1}{2}x$$

Now substitute that in the 2nd equation.

$$y = t^2 + 1$$

$$y = \left(\frac{1}{2}x\right)^2 + 1$$

$$y = \frac{1}{4}x^2 + 1$$

This equation can be more easily graphed.

Example: Find the equation in x and y for the curve

$$x = \frac{1}{t+1} \quad \text{and} \quad y = \frac{3t^2 + 6t + 4}{t^2 + 2t + 1}$$

Solution: Solve the first equation for t.

$$x = \frac{1}{t+1}$$

$$(t+1)x = 1$$

$$tx + x = 1$$

$$tx = 1 - x$$

$$t = \frac{1-x}{x}$$

Substitute into the 2nd equation:

$$y = \frac{3t^2 + 6t + 4}{t^2 + 2t + 1}$$

$$y = \frac{3\left(\frac{1-x}{x}\right)^2 + 6\left(\frac{1-x}{x}\right) + 4}{\left(\frac{1-x}{x}\right)^2 + 2\left(\frac{1-x}{x}\right) + 1}$$

$$y = \frac{\left(\frac{3-6x+3x^2}{x^2}\right) + \left(\frac{6-6x}{x}\right) + 4}{\left(\frac{1-2x+x^2}{x^2}\right) + \left(\frac{2-2x}{x}\right) + 1}$$

Multiply the right side by $\frac{x^2}{x^2}$ to clear the fractions.

$$y = \frac{3 - 6x + 3x^2 + 6x - 6x^2 + 4x^2}{1 - 2x + x^2 + 2x - 2x^2 + x^2}$$

$$y = \frac{x^2 + 3}{1}$$

$$y = x^2 + 3$$

This would be easy to graph.

<u>Finding Parametric Equations</u>

Now we will go the other way. We will either be given what t equals in terms of x, or we can make up our own.

Example: Find a set of parametric equations to represent the graph given by $y = x^2 + 1$ if $t = x$.

<u>Solution</u>: If $t = x$, then substitute t for x in the equation to get the parametric equation for y.

$$y = x^2 + 1$$

$$y = t^2 + 1$$

Therefore, our equations are $x = t$ and $y = t^2 + 1$.

Example: Find a set of parametric equations to represent the graph given by $y = x^2 + 1$ if $t = x + 2$.

Solution: If $t = x + 2$, then solve for x and then substitute for x in the equation to get the parametric equation for y.

$$t = x + 2$$
$$x = t - 2$$

Now substitute.

$$y = x^2 + 1$$
$$y = (t - 2)^2 + 1$$
$$y = t^2 - 4t + 4 + 1$$
$$y = t^2 - 4t + 5$$

Therefore, the parametric equations are

$x = t - 2$ and $y = t^2 - 4t + 5$.