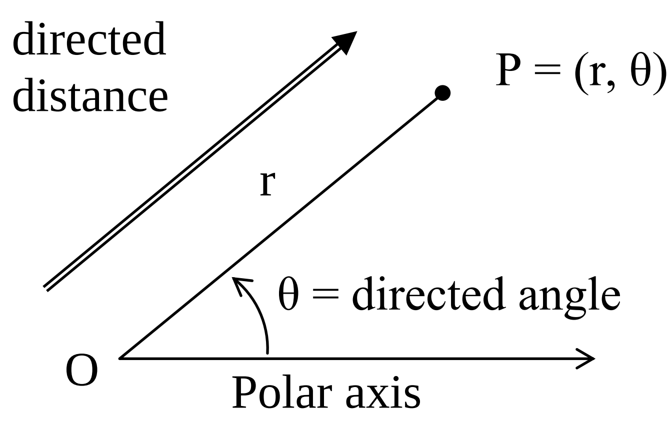
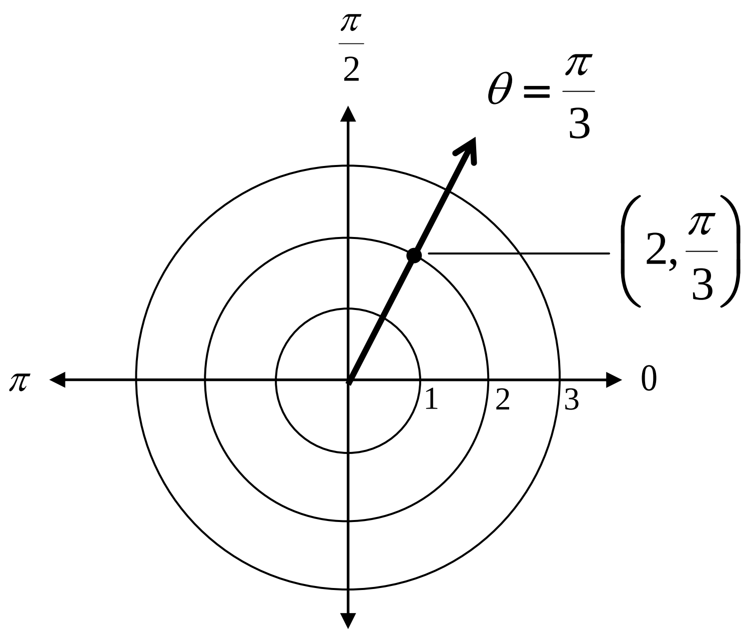
# Polar Coordinates

Familiar: Representing graphs of equations as collections of points $(x, y)$ on the rectangular coordinate system, where $x$ and $y$ represent the directed distances from the coordinate axes to the point $(x, y)$.

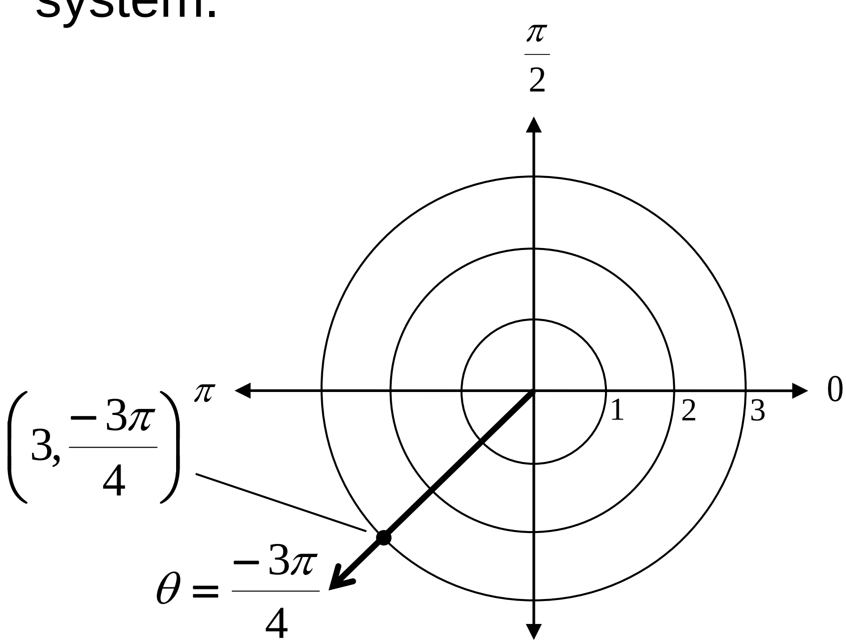
New: The Polar Coordinate System, which consists of a fixed point *O*, called the pole, and a ray, called the polar axis, with its initial point at *O*. Each point *P* in the plane can be labeled with polar coordinates $(r, \theta)$, where *r* is the directed distance from *O* to *P* and θ is an angle in standard position with terminal side at $\overrightarrow{OP}$.



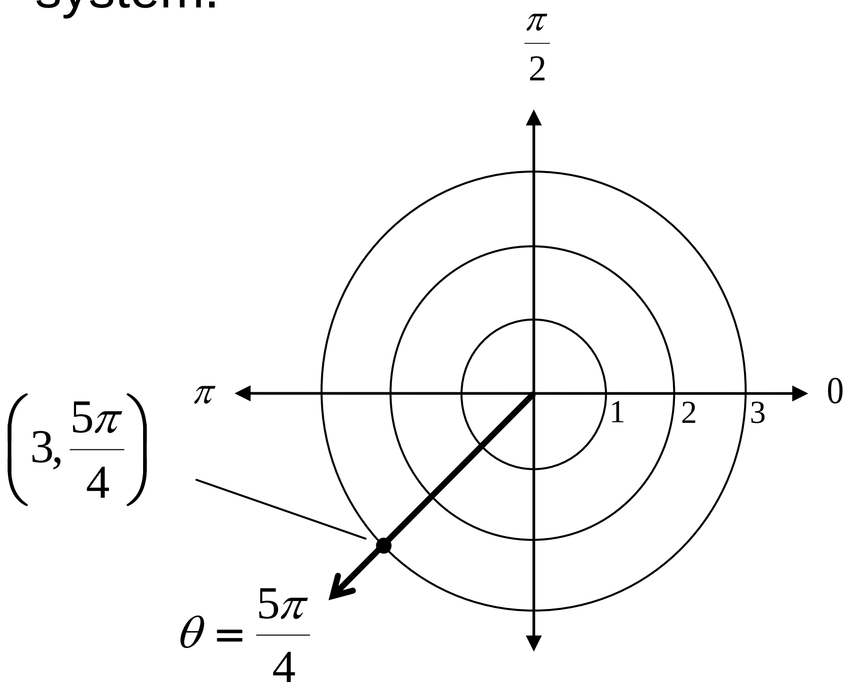
**Example**: Graph the point $\left(2, \frac{\pi}{3}\right)$ in the polar coordinate system.



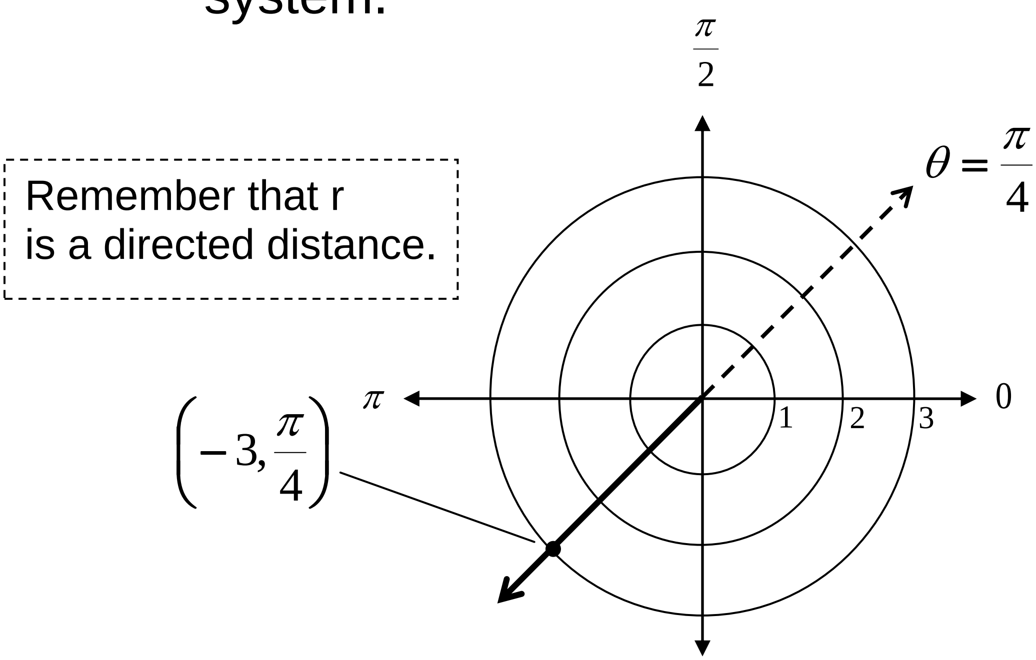

**Example**: Graph the point $\left(3, -\frac{3\pi}{4}\right)$ in the polar coordinate system.



**Example**: Graph the point $\left(3, \frac{5\pi}{4}\right)$ in the polar coordinate system.




**Example**: Graph the point $\left(-3, \frac{\pi}{4}\right)$ in the polar coordinate system.



Notice that the last 3 points graphed the same point. In the polar coordinate system, there are infinitely many coordinates that graph to the same point. This is different than the rectangular coordinate system. In the rectangular coordinate system, each point has a unique representation.

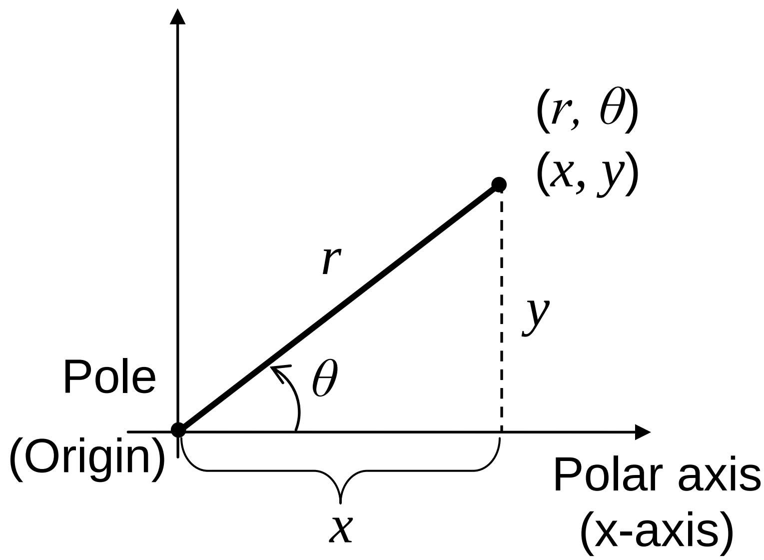
> In general,
>
> $(r,\theta)=(r,\theta\pm 2\pi n)$ or $(r,\theta)=(-r,\theta\pm(2n-1)\pi)$
>
> where $n$ is any integer.

**Example:** Find three other polar representations for the point $\left(2,\frac{\pi}{3}\right)$.

<u>*Solution*</u>: $\left(-2,\frac{4\pi}{3}\right)$, $\left(2,\frac{7\pi}{3}\right)$, $\left(-2,\frac{-2\pi}{3}\right)$, etc

<u>Coordinate Conversion</u>

To see the relationship between polar and rectangular coordinates, let the 2 systems coincide.




Because the point $(x, y)$ lies on the circle of radius $r$, it follows that $r^2 = x^2 + y^2$. You can also see the trig relationships

$$\tan\theta = \frac{y}{x} \qquad \cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

**Coordinate Conversion**

The polar coordinates $(r, \theta)$ are related to the rectangular coordinates $(x, y)$ as follows.

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$\tan\theta = \frac{y}{x} \qquad r^2 = x^2 + y^2$$

**Example**: Convert the point (3, $\pi$) to rectangular coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$x = 3\cos\pi \qquad y = 3\sin\pi$$
$$x = 3(-1) \qquad y = 3(0)$$
$$x = -3 \qquad y = 0$$

The rectangular coordinates are (-3, 0).

**Example**: Convert the point $(4, \frac{\pi}{6})$ to rectangular coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$x = 4\cos\frac{\pi}{6} \qquad y = 4\sin\frac{\pi}{6}$$
$$x = 4\left(\frac{\sqrt{3}}{2}\right) \qquad y = 4\left(\frac{1}{2}\right)$$
$$x = 2\sqrt{3} \qquad y = 2$$

The rectangular coordinates are $(2\sqrt{3},\ 2)$.

**Example**: Convert (-1, -1) to polar coordinates.

$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{-1}{-1}$$
$$\tan\theta = 1$$

According to allsintancos, the tangent = 1 when $\theta$ is in Quadrant I or Quadrant III. Since our point is in quadrant III, we will choose $\theta = \frac{5\pi}{4}$.

Now find r.

$$r^2 = x^2 + y^2$$
$$r^2 = (-1)^2 + (-1)^2$$
$$r^2 = 2$$
$$r = \pm\sqrt{2}$$

Since $\theta$ lies in the same quadrant as $(x,y)$ we need to use the positive value of $r$.

The polar coordinates are $(\sqrt{2}, \frac{5\pi}{4})$.

**<u>Note</u>**: If we had chosen $\frac{\pi}{4}$ for θ, we would have had to use $-\sqrt{2}$ for $r$ to get the same point.

**Example**: Convert (0, -2) to polar coordinates.

*Solution*: This point lies 2 units down on the $y$-axis. That means $r$ = 2 and $\theta = \frac{3\pi}{2}$, giving us the point $(2, \frac{3\pi}{2})$ .

Converting Points Between Polar and Rectangular Systems using a Graphing Calculator

Converting Points from Polar to Rectangular

To convert the point $(r, \theta)$ to rectangular coordinates, do the following:

1. To find the $x$-coordinate, press [$2^{nd}$] [ANGLE] [P▶R$x$(]. Now enter your polar coordinates $r$, $\theta$ and press [ENTER].

2. To find the $y$-coordinate, press [$2^{nd}$] [ANGLE] [P▶R$y$(]. Now enter your polar coordinates $r$, $\theta$ and press [ENTER].

## Converting Points from Rectangular to Polar

To convert the point $(x, y)$ to polar coordinates, do the following:

1. To find the radius, press [2nd] [ANGLE] [R▶P$r$(]. Now enter your rectangular coordinates $x, y$ and press [ENTER]. The value of r will be in decimal form.

2. To find the angle $\theta$, press [2nd] [ANGLE] [R▶P$\theta$ (]. Now enter your rectangular coordinates $x, y$ and press [ENTER]. If you are in DEGREE mode, the angle will be given in degrees. If you are in RADIAN mode, the answer will be given in radians.

*Note: The polar coordinates of a given point $(x, y)$ are not unique. There are other possible answers.

## Equation Conversion

To convert rectangular equations to polar equations, simply replace $x$ with $r\cos\theta$ and $y$ with $r\sin\theta$. Converting polar equations to rectangular equations requires considerable ingenuity.

**Example**: Convert the rectangular equation $x = y^2$ to a polar equation.

*Solution*: Use $x = r\cos\theta$ and $y = r\sin\theta$ and substitute.

$$x = y^2$$

$$(r\cos\theta) = (r\sin\theta)^2$$

$$r\cos\theta = r^2\sin^2\theta$$

$$\frac{r\cos\theta}{r} = \frac{r^2\sin^2\theta}{r}$$

$$\cos\theta = r\sin^2\theta$$

$$\frac{\cos\theta}{\sin^2\theta} = \frac{r\sin^2\theta}{\sin^2\theta}$$

$$\boxed{r = \cot\theta\csc\theta}$$

**Example**: Convert the polar equation $r = 3$ to a rectangular equation.

*Solution*: Think about what this equation looks like. It is all the points that are 3 units from the pole. This is a circle of radius 3. In rectangular form, this equation would be $x^2 + y^2 = 9$.

**Example**: Convert the polar equation $\theta = \frac{\pi}{6}$ to a rectangular equation.

*Solution*: Think about what this equation looks like. It is all of the points that lie on the line that makes an angle of $\frac{\pi}{6}$ with the positive polar axis.

We know that $\tan\theta = \frac{y}{x}$, so we get

$$\tan\theta = \frac{y}{x}$$

$$\tan\frac{\pi}{6} = \frac{y}{x}$$

$$\frac{\sqrt{3}}{3} = \frac{y}{x}$$

$$\boxed{y = \frac{\sqrt{3}}{3}x}$$

This answer fits what we would expect. We should get a linear equation of the form $y = mx$, since this line goes through the origin with positive slope.

**Example**: Convert the polar equation $r = \sec\theta$ to a rectangular equation.

*Solution*: What we have to work with is

$$x = r\cos\theta \qquad \tan\theta = \frac{y}{x}$$
$$y = r\sin\theta \quad \text{and} \quad r^2 = x^2 + y^2$$

Since secant and cosine are related, we start with $r = \sec\theta$.

$$r = \sec\theta$$
$$r = \frac{1}{\cos\theta}$$
$$r\cos\theta = 1$$

We know that $x = r\cos\theta$, we must have the equation

$$x = 1$$

This is our rectangular equation.
It is a vertical line at $x = 1$.

**Example**: Convert the polar equation $r^2 = \dfrac{1}{1-\sin 2\theta}$ to a rectangular equation.

Solution: Clear the fraction and work with the equation until something useful or familiar comes up.

$$r^2 = \frac{1}{1-\sin 2\theta}$$
$$r^2(1-\sin 2\theta) = 1$$
$$r^2 - r^2\sin 2\theta = 1$$

We know that $\sin 2\theta = 2\sin\theta\cos\theta$ so substitute.

$$r^2 - r^2\sin 2\theta = 1$$
$$r^2 - r^2(2\sin\theta\cos\theta) = 1$$
$$r^2 - r^2(2\sin\theta\cos\theta) = 1$$

There is an $r\cos\theta$ and $r\sin\theta$ embedded here.

$$r^2 - 2(r\sin\theta)(r\cos\theta) = 1$$

Substitute for $r\cos\theta$, $r\sin\theta$, and $r^2$ to get

$$\boxed{x^2 + y^2 - 2xy = 1}$$