Quadratic Functions

Look at: $f(x) = 4x-7$
$f(x) = 3$
$f(x) = x^2 + 4$

*These are all examples of polynomial functions.

Definition: Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. A polynomial function of x with degree n is the function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1x + a_0$$

**Polynomial functions are classified by degree. The degree of a polynomial function is equal to the highest exponent on x.

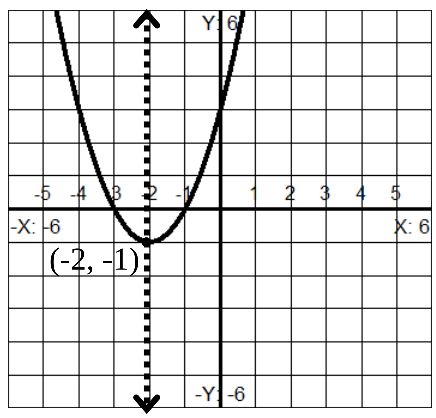
Example: $f(x) = 7x^5 + 2x^2 + 1$ has degree 5
$f(x) = 4x-3$ has degree 1
$f(x) = 13$ has degree 0 $(13 = 13x^0)$

**2nd degree polynomials are called quadratic functions.

Definition: Let Let a, b, and c be real numbers with $a \neq 0$. A quadratic function is the function $f(x) = ax^2 + bx + c$.

The graph of a quadratic function is a special "U"-shaped curve called a parabola .

Graph $y = x^2 + 4x + 3$



*All parabolas are symmetric with respect to a line called the axis of symmetry or simply the axis of the parabola. The point where the axis intersects the parabola is the vertex.

Example: Graph $f(x) = x^2 – 8x +17$

[TRACE] to find the approximation of the vertex (4, 1).

*The x-coordinate of the vertex always tells us the axis of symmetry. In this case it is $x = 4$.

Review Transformations:

Consider $f(x) = x^2$

What would the equation be if you shift this graph 3 units to the left?

$f(x) = (x+3)^2$ *This is $f(x-c)$.*

Now shift your new graph 2 units down. What is the equation?

$f(x) = (x+3)^2 - 2$ *This is $f(x) – c$*

What is the vertex and axis of symmetry?

vertex (-3, -2) and axis of symmetry $x = -3$

Consider $f(x) = x^2$

Cause a vertical stretch of 3. What is the equation?

$f(x) = 3x^2$ *This is $cf(x)$.*

Reflect this in the x-axis. What is the equation?

$f(x) = -3x^2$ *This is $-f(x)$.*

How does the graph of $f(x) = -3x^2$ compare to $f(x) = x^2$?

It's narrower and flipped up-side-down.

Example: Look at $f(x) = -3(x+3)^2 - 2$. Describe the graph.

This is a parabola with vertex (-3, -2) which opens downward and is narrower than $f(x) = x^2$ *by a factor of 3. The Axis of symmetry is x = -3. Because it opens downward, the vertex (-3, -2) is the maximum y-value on the graph.*

Definition: The standard form of a quadratic function is

$$f(x) = a(x - h)^2 + k, \ a \neq 0$$

The vertex is (h, k) and the axis of symmetry is x=h.

*If a is **positive**, the parabola opens **upward**.
*If a is **negative**, the parabola opens **downward**.

*The larger |a| is, the more narrow (steep) the parabola is.

Completing the Square

To write a quadratic function in standard form we use the process of completing the square.

Example: Write $f(x) = x^2 - 6x – 2$ in standard form.

We need the form $f(x) = (x – h)^2 + k$.

1. Since $x^2 – 6x – 2$ is not a perfect square trinomial, we must **make** it one. Slide the -2 over and put in a "+ ____" (blank) so we can pick the number that will give us a perfect square trinomial. We must also put a "- ___" (blank) on the same side of the equation so that we are adding and subtracting the same amount, thus not changing the value of the original function.

   $f(x) = x^2 – 6x + \_\_\_ – 2 – \_\_\_$

2. Put ( ) around the first 3 terms to group them.

   $f(x) = (x^2 – 6x + \_\_\_) – 2 – \_\_\_$

3. Decide what number to put in the blank so that you will be able to factor the trinomial into something of the form

$(x - h)^2$. (This is a perfect square trinomial.) To do this, take half of the coefficient of the linear term (the term with the x) and square it.

Half of 6 is 3. We square the 3 to get 9. This goes in both blanks. (i.e. ½(6) = 3 and $3^2 = 9$)

$$f(x) = (x^2 - 6x + \underline{\;9\;}) - 2 - \underline{\;9\;}$$

4. Now we can factor using backwards FOIL.

$$f(x) = (x - 3)(x - 3) - 11$$
$$f(x) = (x - 3)^2 - 11$$

<u>Note</u>: *The number in the (   ) will always be half of the coefficient of x in the original function.*

The vertex is (3, -11)
The axis of symmetry is x = 3

Example: Write in standard form.

(1) $f(x) = x^2 + 8x + 9$

$f(x) = (x^2 + 8x + \_\_\_) + 9 - \_\_\_$
$f(x) = (x^2 + 8x + \underline{16}) + 9 - \underline{16}$ *$\frac{1}{2}(8) = 4 \Rightarrow 4^2 = 16$*
$f(x) = (x + 4)^2 - 7$

(2) $f(x) = 2x^2 + 8x + 7$

$f(x) = (2x^2 + 8x + \_\_\_) + 7 - \_\_\_$
$f(x) = 2(x^2 + 4x + \_\_\_) + 7 - \_\_\_$ *Factor out the 2.*
$f(x) = 2(x^2 + 4x + \underline{4}) + 7 - \underline{8}$ *$\frac{1}{2}(4) = 2 \Rightarrow 2^2 = 4$*
(We really added $2 \cdot 4 = 8$, so we must subtract 8.)
$f(x) = 2(x + 2)^2 - 1$

Example: Sketch the graph of $f(x) = x^2 + 2x - 8$ and identify the vertex, axis, and *x*-intercepts of the parabola. Also find the y-intercepts.

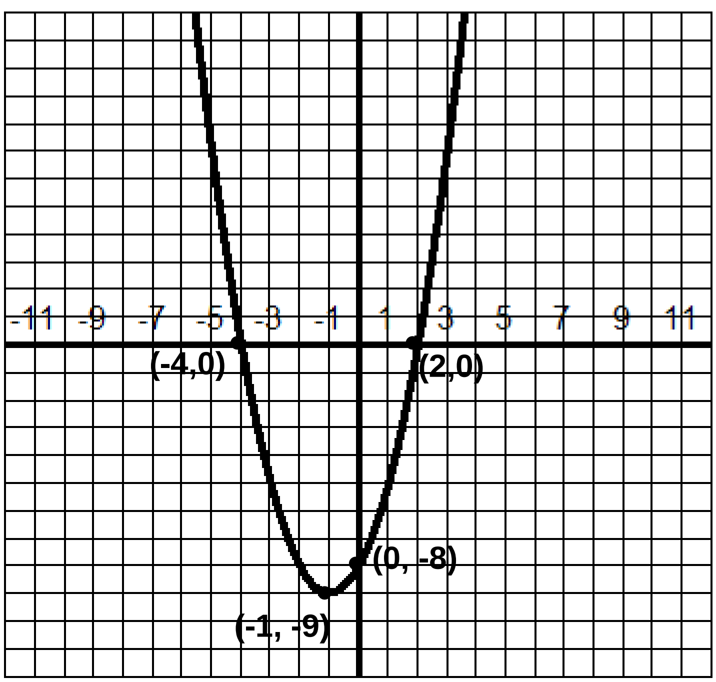
$f(x) = (x^2 + 2x + \underline{1}) - 8 - \underline{1}$
$f(x) = (x + 1)^2 - 9$
Vertex (-1, -9)
Axis of symmetry: x = -1

For the x-intercepts, let f(x) = 0 and solve.

$0 = x^2 + 2x - 8$
$0 = (x + 4)(x - 2)$
$x + 4 = 0$ *or* $x - 2 = 0$
$x = -4$ *or* $x = 2$*, so the x-intercepts are (-4,0) and (2,0)*

For the y-intercepts, let x = 0 and solve.

$f(0) = 0^2 + 2(0) - 8$
$f(0) = -8$*, so the y-intercept is (0, -8)*



The parabola opens upward and it is the same shape as the standard graph of $f(x) = x^2$.

Example: Find the vertex of $f(x) = -2x^2 - 4x + 1$.

$f(x) = (-2x^2 - 4x \ + \underline{\quad\quad}) + 1 - \underline{\quad\quad}$

$f(x) = -2(x^2 + 2x \ + \underline{\quad\quad}) + 1 - \underline{\quad\quad}$ *Watch negatives!*

$f(x) = -2(x^2 + 2x \ + \underline{1}) + 1 - \underline{(-2)}$ *Watch negatives!*

$f(x) = -2(x^2 + \underline{1})^2 + 3$

The vertex is (-1, 3).

Example: Find the standard form of the equation of the parabola that has vertex at (1, -2) and passes through the point (3, 6).

From the vertex, we have this much of the equation:

$$f(x) = a(x - 1)^2 - 2.$$

To find *a*, we substitute the point (3, 6) and solve for *a*.

$$6 = a(3 - 1)^2 - 2$$
$$6 = a(2)^2 - 2$$
$$6 = 4a - 2$$
$$8 = 4a$$
$$a = 2$$

So, the equation must be $f(x) = 2(x - 1)^2 - 2$.

Applications

Many applications involve finding the minimum or maximum value of a quadratic function. Some are difficult to write in standard form. Here is an alternative way to find the vertex.

Definition: The vertex of the graph $f(x) = ax^2 + bx + c$ is

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

Example: Find the vertex of the parabola defined by

$$f(x) = 3x^2 - 11x + 16$$

By Completing the Square:

$$f(x) = (3x^2 - 11x + \_\_\_\_) + 16 - \_\_\_\_$$
$$f(x) = 3(x^2 - \frac{11}{3}x + \frac{121}{36}) + 16 - \frac{121}{36}$$
$$f(x) = 3(x - \frac{11}{6})^2 + \frac{71}{12}$$

By the alternative method: $f(x) = 3x^2 - 11x + 16$

a = 3	b = -11	c = 16

$$x\text{- coordinate} = \frac{-b}{2a} = \frac{-(-11)}{2(3)} = \frac{11}{6}$$

$$y\text{-coordinate} = f\left(\frac{-b}{2a}\right) = f(\frac{11}{6}) = 3(\frac{11}{6})^2 - 11(\frac{11}{6}) + 16 = \frac{71}{12}$$

Example: find the vertex of $f(x) = -0.0032x^2 + x + 3$.

a = -0.0032	b = 1	c = 3

x-coordinate = $\frac{-1}{2(-0.0032)}$ = *156.25*

y-coordinate = $-0.0032(156.25)^2 + (156.25) + 3 = 81.125$