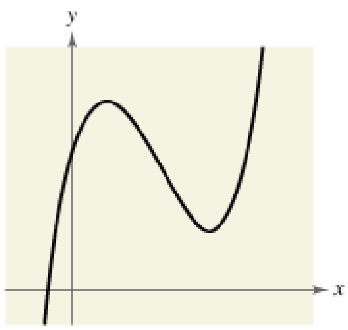
Polynomial Functions of Higher Degree

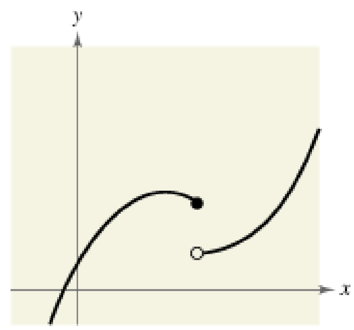
Graphs of Polynomial Functions

2 Basic Features of Polynomial Functions:

1. They are continuous. This means there are no breaks, gaps, or holes.

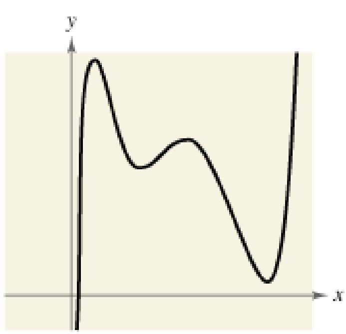


polynomial function

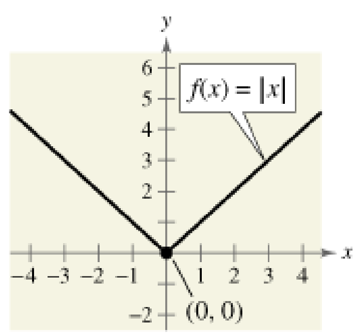


not a polynomial function

2. They have smooth, rounded turns and bends.



polynomial function



not a polynomial function

Question: Which function have we studied that is not continuous, and thus is not a polynomial function?

The Greatest Integer Function

Question: Which function have we studied that does not have smooth and rounded bends and turns, and thus is not a polynomial function?

The Absolute Value Function

<u>Power Functions</u>

Graph $y = x^2$ using the window $-2 < x < 2$ and $0 < y < 4$

Graph $y = x^4$ on the same graph.

Graph $y = x^{20}$ on the same graph.

Question: What happens to the graph as the exponent gets larger?

It flattens out at the origin.

Graph $y = x^3$ using the window $-2 < x < 2$ and $-2 < y < 2$

Graph $y = x^7$ on the same graph.
Graph $y = x^{21}$ on the same graph.

<u>Question:</u> What happens to the graph as the exponent gets larger?

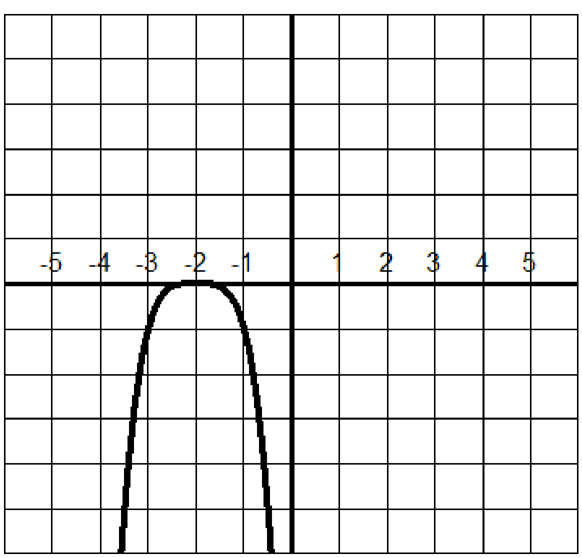
It flattens out at the origin.

<u>Power Functions</u>:

- Even Power Functions: $f(x) = x^n$ where *n* is <u>even</u>.
- Odd Power Functions: $f(x) = x^n$ where *n* is <u>odd</u>.

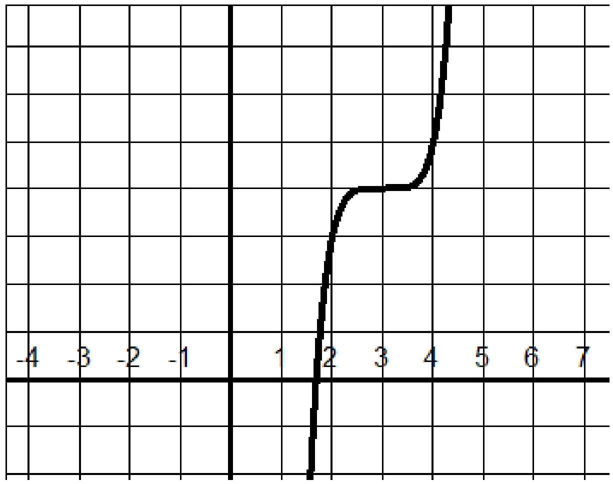
Both graphs flatten out around the origin as *n* increases.

Example: Sketch the graph of $f(x) = -(x + 2)^4$ on your calculator.



This is a reflection of $f(x) = x^4$ over the x-axis and a shift of 2 units to the left.

Example: Graph $f(x) = (x-3)^5 + 4$ on your calculator.

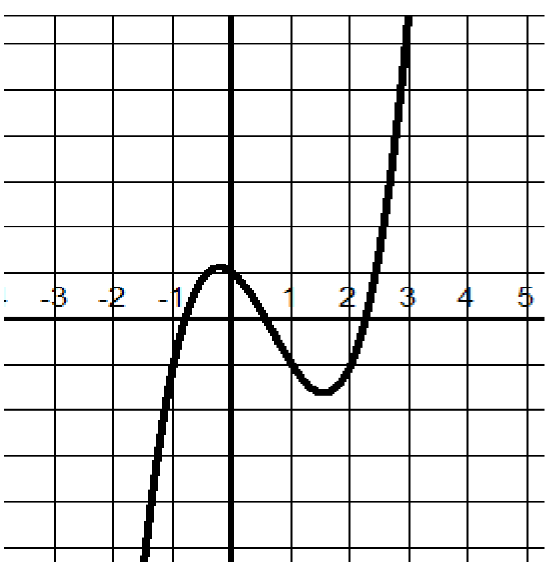


Describe the transformation on $f(x) = x^5$ to get this graph.

Shift the graph 3 units right and 4 units up.

<u>The Leading Coefficient</u>

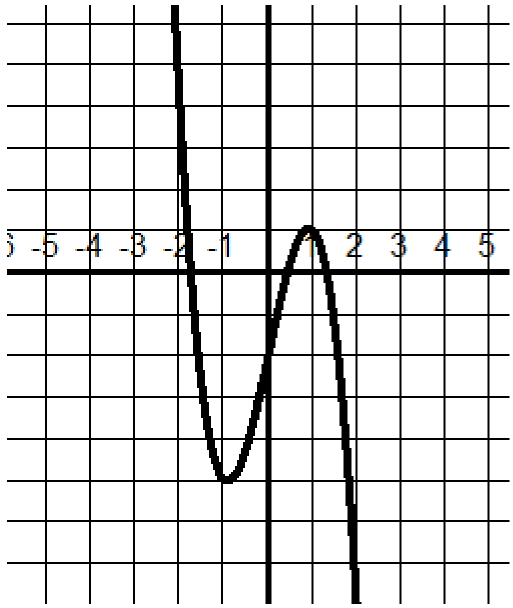
Look at the graph of $f(x) = x^3 – 2x^2 – x + 1$



We want to look at what happens to the graph as $x \to$ **4** and $x \to$ - **4**.

This graph ***falls*** to the left (as $x \to$ - **4**) and ***rises*** to the right (as $x \to$ **4**).

Look at the graph of $f(x) = -2x^3 + 5x - 2$.



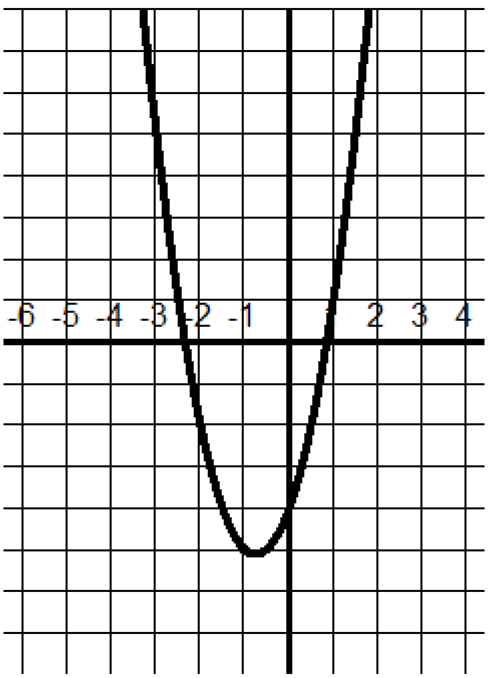
Describe what the graph does $x \to 4$ and $x \to -4$.

It rises to the left and falls to the right.

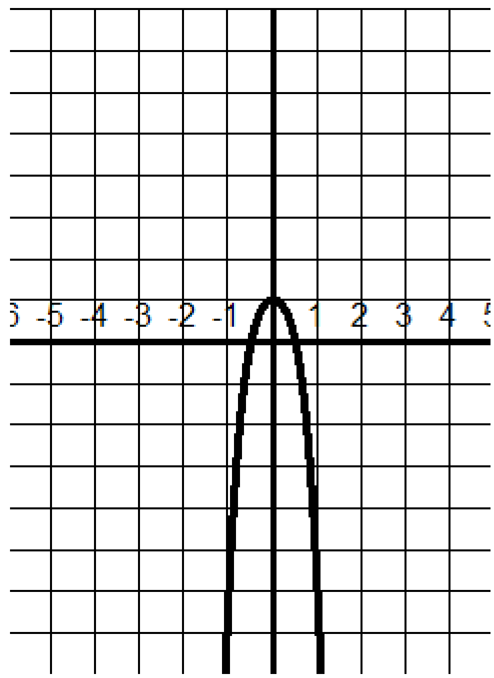
****<u>Note</u>**: All odd degree functions will behave this way. If the leading coefficient is positive, the graph falls to the left and rises to the right. If the leading coefficient is negative, the graph rises to the left and falls to the right.

Look at the following graphs:

$f(x) = 2x^2 + 3x - 4$



$f(x) = -4x^4 - 3x^2 + 1$



Describe what the graph does $x \to 4$ and $x \to -4$.

The first graph rises to the left and the right, and the second graph falls to the left and to the right.

**__Note__: All even degree functions will behave this way. If the leading coefficient is positive, the graph rises to the left and to the right. If the leading coefficient is negative, the graph falls to the left and to the right.

Leading Coefficient Test:

$f(x) = a_n x^n + \ldots$	a_n *is positive*	a_n *is negative*
n is even	$\nwarrow \quad \nearrow$	$\swarrow \quad \searrow$
n is odd	$\swarrow \quad \nearrow$	$\nwarrow \quad \searrow$

Examples: Find the right-hand and left-hand behaviors.

(a) $f(x) = x^4 + 7x^3 - 14x - 9$

rises to the left and right

(b) $f(x) = -x^5 + 2x^3 - 14x^2 + 6$

rises to the left and falls to the right

Example: $f(x) = 1 - 3x^2 - 4x^6$

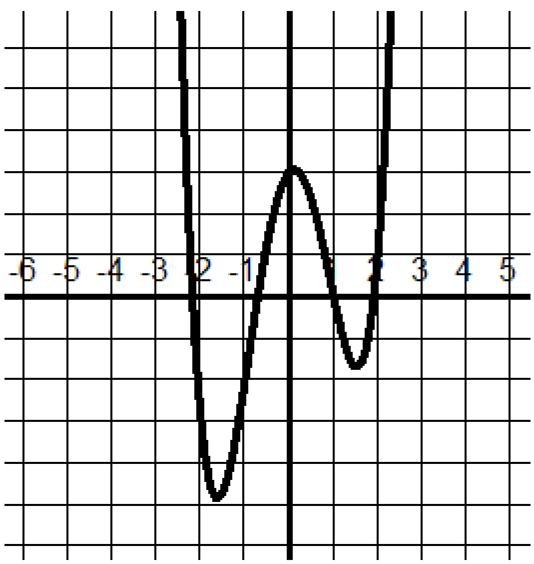
falls to the left and right

Zeros of Polynomial Functions

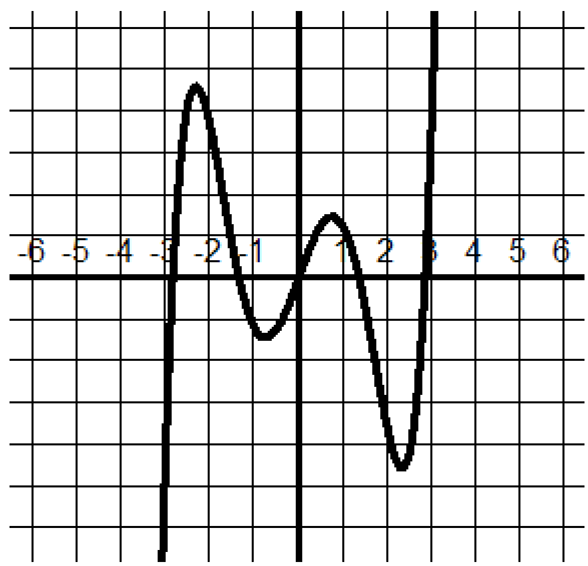
**The leading coefficient tells us whether the graph eventually rises or falls to the right or left. Other characteristics of the graph, such as intercepts and min/max points must be determined by other tests.

Look at the following graphs:

$f(x) = x^4 - 5x^2 + x + 3$



$f(x) = \frac{1}{5}x^5 - 2x^3 + 3x$



How many turning points does each have?

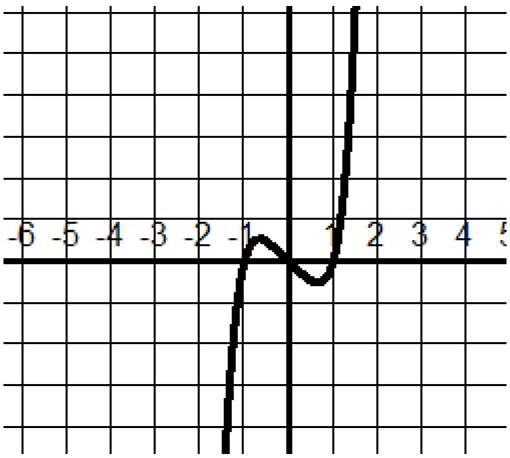
3 and 4, respectively

How does this number relate to the exponent on the leading term?

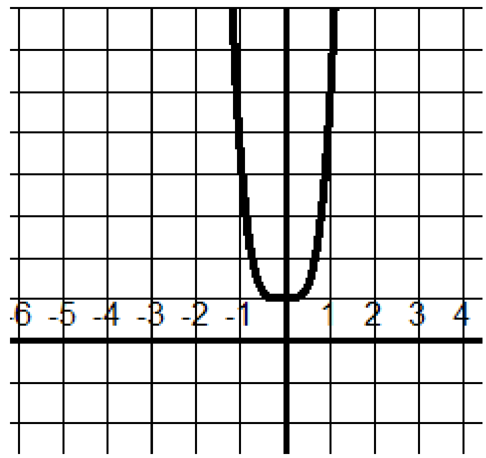
1 less

Look at the following graphs:

$f(x) = x^5 - x$



$f(x) = 4x^4 + x^3 + 1$



How many turning points does each have?

2 and 1, respectively

****The most number of turning points for a polynomial function of degree *n* is *n-1*.**

Look at the 4 graphs again. How many zeros does each have?

4, 5, 3, 0 respectively

****The most real zeros that a polynomial function of degree n has is n.**

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, then the following statements are equivalent.

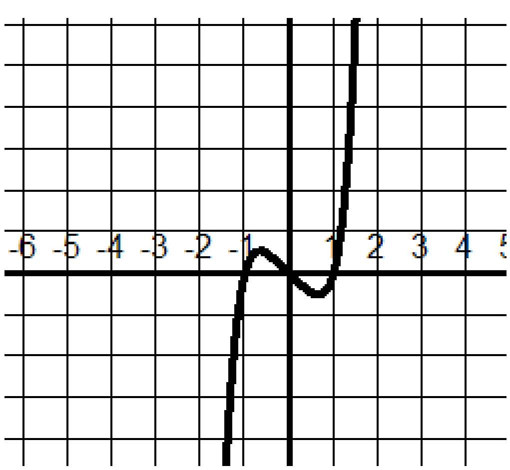
1. $x = a$ is a zero of f.
2. $x = a$ is a solution of the equation $f(x) = 0$.
3. $(x - a)$ is a factor of $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

Example: Find the zeros of $f(x) = x^5 - x$

Set f(x) = 0 and solve.

$$x^5 - x = 0$$
$$x(x^4 - 1) = 0$$
$$x(x^2 - 1)(x^2 + 1) = 0$$
$$x(x + 1)(x - 1)(x^2 + 1) = 0$$

$x = 0$, $x = -1$, and $x = 1$ are the real zeros
(note that $x^2 + 1$ would give us complex zeros)

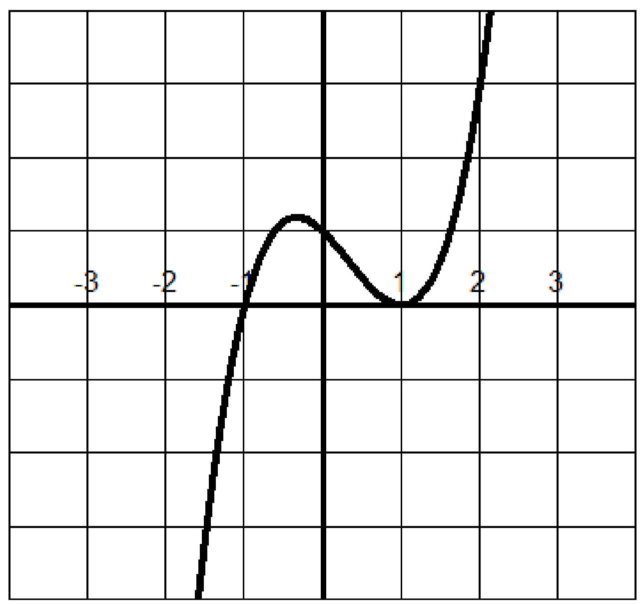


Example: Find all of the zeros of $f(x) = x^3 - x^2 - x + 1$

$f(x) = x^2(x - 1) - (x - 1)$ (factor by grouping)
$f(x) = (x^2 - 1)(x - 1)$
$f(x) = (x + 1)(x - 1)(x - 1)$

Let $f(x) = 0$ and solve.
$0 = (x + 1)(x - 1)(x - 1)$, so $x = -1$ or $x = 1$ are the zeros

Look at the graph:



What happens at $x = 1$?

The graph only <u>touches</u> the x-axis.

Do you notice anything about the zero $x = 1$ in our factored equation?

$x = 1$ is a repeated zero because $(x-1)$ is a factor twice.

Repeated Zeros:

A factor $(x - a)^k$, where $k > 1$, yields a repeated zero $x = a$ of multiplicity k.

1. If k is odd, the graph crosses the x-axis at x = a.
2. If k is even, the graph touches the x-axis at x = a (but does not cross it).

Example: If a polynomial function *f* has a repeated zero $x = 3$ with multiplicity 4, the graph of *f* touches the *x*-axis at $x = 3$.

If a polynomial function *f* has a repeated zero $x = 4$ with multiplicity 3, the graph of *f* crosses the *x*-axis at $x = 4$.

Example: Sketch the graph of $f(x) = \frac{1}{4}x^4 - 2x^2 + 3$

1. Since the leading coefficient is positive, we know that the graph opens up to the left and right.

2. Find the zeros by letting f(x) = 0.
   $0 = \frac{1}{4}x^4 - 2x^2 + 3$
   $0 = x^4 - 8x^2 + 12$ (Mult. through by 4)
   $(x^2 - 6)(x^2 - 2) = 0$
   $x^2 - 6 = 0$ or $x^2 - 2 = 0$
   $x^2 = 6$ or $x^2 = 2$
   $x = \pm\sqrt{6}$ (**≈±**2.4) or $x = \pm\sqrt{2}$ (**≈±**1.4)

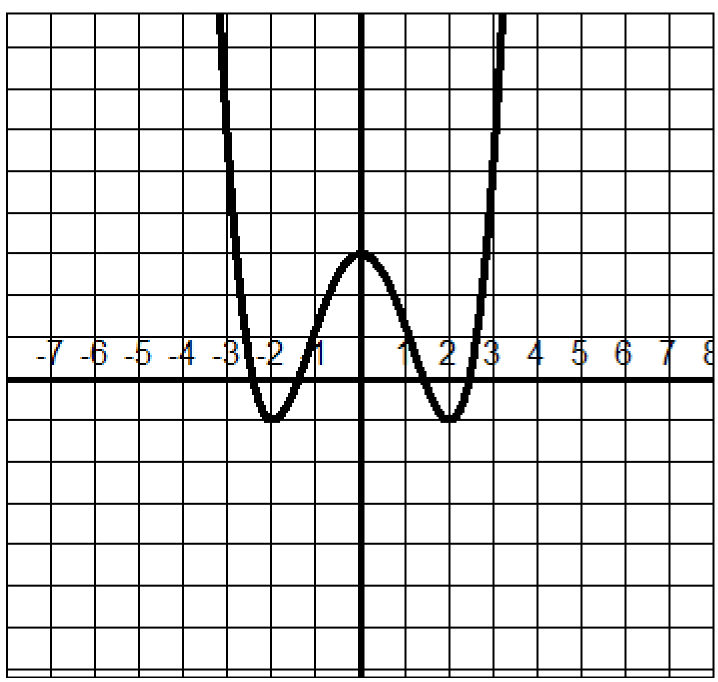
There are no repeating zeros, so the graph crosses the x-axis at approximately "2.4 and "1.4.

3. Find a few key points. This would include the y-intercept (let x = 0) and some point between the zeros to get an idea of how high or low the bends go.

x	f(x)
-2	-1
0	3
2	-1

Note that since the equation is even, 2 and -2 give the same value for f(x).

4. Sketch the graph.



Example: Sketch the graph of $f(x) = x^3 – 2x^2$.

1. The leading coefficient is positive, so the graph rises to the right and falls to the left.

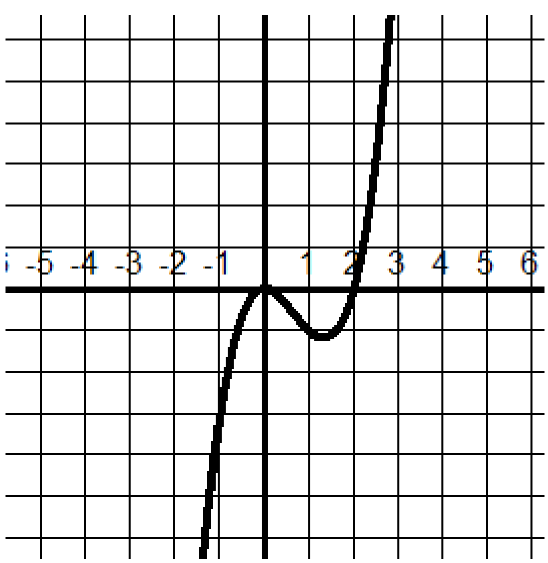
2. Find the zeros.

   $0 = x^3 – 2x^2$
   $0 = x^2(x – 2)$
   $x = 0$ or $x = 2$
   0 has multiplicity of 2, so the graph only touches at 0.

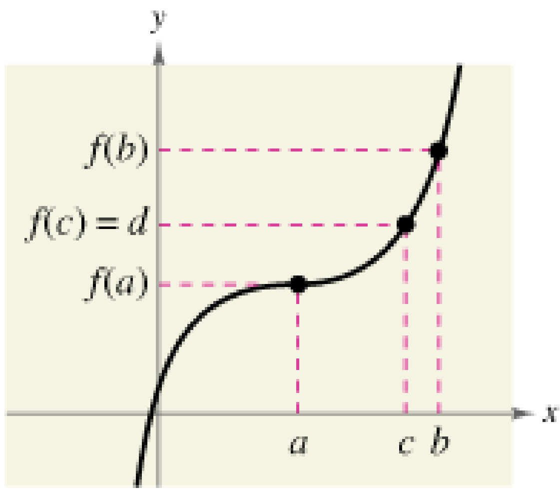
3. Find a few key points.

x	f(x)
-1	-3
0	0
1	-1

4. Sketch the graph.



Intermediate Value Theorem



Let *a* and *b* be real numbers such that $a < b$. If *f* is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, *f* takes on every value between $f(a)$ and $f(b)$.

**If $a < b$, and *c* is between and *a* and *b*, then $f(x)$ must be between $f(a)$ and $f(b)$.

<u>Question</u>: How does the Intermediate Value Theorem help us graph polynomial functions?

If you can find a value x = a at which f is positive and another value x = b at which f is negative, you can conclude that f has at least one real zero between a and b.

Example: Use the Intermediate Value Theorem and your graphing calculator to approximate the real zeros of $f(x) = 4x^3 - 7x^2 - 21x + 18$.

1. Enter the equation into [y=].
2. Look at the graph to visually get an idea of how many/where the zeros are. In this case there are 3 zeros and they are between -3 and 3.
3. Put the values into a table:
   [TBLSET]
   TblStart = -3
   ΔTbl = 1
   [TABLE]

   Look at the table. Notice that you have zeros at x = -2 and x = 3. Also, since the y-values change from positive to negative between x = 0 and x = 1, then by the Intermediate Value Theorem, we know that there is a real zero on the interval [0,1].

4. Go back to [TBLSET] and start your table at 0, using ΔTbl = 0.1. [TABLE] shows us that the zero is between x = 0.7 and x = 0.8. Repeat this process to get a closer approximation of the zero. Start your table at 0.7 and use)Tbl = 0.01.

Answer: The zeros are -2, 3, and approximately 0.75.