Complex Numbers

<u>The Imaginary Unit i</u>

Solve $x^2 = -1$

*We can't solve this because there is no real number that we can square to get -1.

*Mathematicians created a system to overcome this problem!

<u>Definition</u>: The <u>imaginary unit</u> is defined as $i = \sqrt{-1}$ where $i^2 = -1$.

*This gives us a whole new set of numbers called the <u>complex numbers</u>.

<u>Definition</u>: If *a* and *b* are real numbers, the number $a + bi$, where the number *a* is called the *real part* and the number bi is called the *imaginary part*, is a <u>complex number</u> written in standard form.

Look at $a + bi$.

If $b = 0$, then we just have *a*, which is a <u>real number</u>.
If $b \neq 0$, we have an <u>imaginary number</u>.
If $a = 0$, we just have bi, which is called a <u>pure imaginary number</u>.

Examples: $4 + 0i$ real number
$7 - 3i$ imaginary number
$0 + 6i$ pure imaginary number

*The set of complex numbers consists of the set of real numbers and the set of imaginary numbers.

Equality of Complex Numbers: Two complex numbers $a + bi$ and $c + di$ are equal to each other if and only if $a = c$ and $b = d$.

Example: If $a + bi = 13 + 4i$, then a = 13 and b = 4.

If $(a+6) + 2bi = 16 - 4i$, then a+6 = 16 and 2b = -4.
So, a = 10 and b = -2.

Operations on Complex Numbers

To add two complex numbers, add the two real parts and then add the two imaginary parts.

That is, $(a + bi) + (c + di) = (a + c) + (b + d)i$

To subtract two complex numbers, subtract the two real parts and then subtract the two imaginary parts.

That is, $(a + bi) - (c + di) = (a - c) + (b - d)i$

Examples: Find the sum or difference.

(a) $(4+7i) + (6 - 2i)$ answer: $10 + 5i$

(b) $(6 - 3i) + (1 + 2i)$ answer: $7 - i$

(c) $(5 - i) - (2 - 4i)$ answer: $3 + 3i$

(d) $(5 - 6i) - (3 - 2i) + 4i$ answer: 2

Definition: The additive identity of the complex numbers is zero. The additive inverse of $a + bi$ is $-(a + bi) = -a - bi$.

Example: The additive inverse of $3 + 4i$ is $-3 - 4i$.
The additive inverse of $5 - 6i$ is $-5 + 6i$.

*Additive inverses should add to give the additive identity.

Properties of Complex Numbers: Many of the properties of real numbers are valid for complex numbers. Here are some:

- Associative Property of Addition and Multiplication
- Commutative Property of Addition and Multiplication
- The Distributive Property
- The Additive Inverse Property
- The Identity Property of Addition

Multiples of i:

$i^1 = i$	$i^1 = i$
$i^2 = -1$	$i^2 = -1$
$i^3 = i^2 \bullet i^1 = -1i = -i$	$i^3 = -i$
$i^4 = i^2 \bullet i^2 = (-1)(-1) = 1$	$i^4 = 1$
$i^5 = i^4 \bullet i^1 = (1)i = i$	$i^5 = i$
$i^6 = i^4 \bullet i^2 = (1)(-1) = -1$	$i^6 = -1$
$i^7 = i^4 \bullet i^3 = (1)(-i) = -i$	$i^7 = -i$
$i^8 = i^4 \bullet i^4 = (1)(1) = 1$	$i^8 = 1$
$i^9 = i^4 \bullet i^4 \bullet i^1 = (1)(1)(i) = i$	$i^9 = i$
$i^{10} = i^4 \bullet i^4 \bullet i^2 = (1)(1)(-1) = -1$	$i^{10} = -1$
$i^{11} = i^4 \bullet i^4 \bullet i^3 = (1)(1)(-i) = -i$	$i^{11} = -i$
$i^{12} = i^4 \bullet i^4 \bullet i^4 = (1)(1)(1) = 1$	$i^{12} = 1$
etc.	*etc.*

*To simplify i^n, take out all multiples of i^4.

*To multiply complex numbers, use Distributive or FOIL.

Example: $(a + bi)(c + di) = ac + adi + bci + bdi^2$
$= ac + adi + bci + bd(-1)$
$= (ac - bd) + (bc + bd)i$

Example: $(2 + 3i)(4 - 7i) = 8 - 14i + 12i - 21i^2$
$= 8 - 2i + 21$
$= 29 - 2i$

Examples: Multiply.

(a) $2(-5+17i)$ answer: $-10+34i$

(b) $(3 - i)(4+5i)$ answer: $17+11i$

(c) $(1 + 7i)^2$ answer: $1 + 14i + 49i^2 = -48 + 14i$

(d) $(4 + 5i)(4 – 5i)$ answer: 41

(e) $(5 – 6i)(3 – 2i)$ answer: $3 – 28i$

Complex Conjugates

*Remember that we cannot ever leave a radical in the denominator of a fraction. We must always rationalize the denominator.

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \bullet \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$\frac{3}{\sqrt{12}} = \frac{3}{\sqrt{12}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{36}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

$$\frac{3}{4-\sqrt{5}} = \frac{3}{4-\sqrt{5}} \bullet \frac{4+\sqrt{5}}{4+\sqrt{5}} = \frac{12+3\sqrt{5}}{16-5} = \frac{12+3\sqrt{5}}{11}$$

$4+\sqrt{5}$ and $4-\sqrt{5}$ are called <u>conjugates</u>.

Look at $\frac{2}{3+4i}$ Since the denominator contains a radical (i), we must get it out of the denominator.

Use the conjugate of $3 + 4i \rightarrow 3 - 4i$

$$\frac{2}{3+4i} \bullet \frac{(3-4i)}{(3-4i)} = \frac{6-8i}{3+16} = \frac{6-8i}{19} = \frac{6}{19} - \frac{8}{19}i$$

↑ Standard Form

<u>Definition</u>: Two complex numbers of the forms $a+bi$ and $a-bi$ are called <u>complex conjugates</u>. Their product is a real number.

Examples: Write the following quotients in standard form.

(a) $\frac{2}{1-i}$ answer: $1 + i$

(b) $\frac{2-i}{4+3i}$ answer: $\frac{1}{5} - \frac{2}{5}i$

Example: Divide (1 + i) by (2 - i). Write the result in standard form.

$$\frac{1+i}{2-i} = \frac{1+i}{2-i} \cdot \frac{(2+i)}{(2+i)} = \frac{2+3i-1}{4+1} = \frac{1+3i}{5} = \frac{1}{5} + \frac{3}{5}i$$

Complex solutions of quadratic Equations

Definition: The principal square root of a negative number is defined by $\sqrt{-a} = \sqrt{a} \bullet \sqrt{-1} = \sqrt{a}\, i$ where a is positive.

$\sqrt{-3} = \sqrt{3}\, i$
$\sqrt{-10} = \sqrt{10}\, i$
$\sqrt{-25} = 5i$
$\sqrt{-12} = 2\sqrt{3}\, i$

Examples: Simplify.

(a) $\sqrt{-24}$ answer: $2\sqrt{6}\, i$

(b) $\sqrt{-36}$ answer: $6i$

(c) $4 + \sqrt{-20}$ answer: $4 + 2\sqrt{5}\,i$

(d) $(\sqrt{-2})^2$ answer: -8

Be careful when multiplying!

$$\sqrt{-2} \bullet \sqrt{-3} \neq \sqrt{(-2)(-3)}$$
$$\sqrt{-2} \bullet \sqrt{-3} \neq \sqrt{6}$$

**You must factor out the $\sqrt{-1}$ first!

$$\sqrt{-2} \bullet \sqrt{-3} = \sqrt{2}\,i \bullet \sqrt{3}\,i = \sqrt{6}\,i^2 = -\sqrt{6}$$

Examples: Multiply.

(a) $\sqrt{-8} \bullet \sqrt{-6}$ answer: $-4\sqrt{3}$

(b) $(1 - \sqrt{-14})^2$ answer: $-13 - 2\sqrt{14}\,i$

(c) $(5 - \sqrt{-4})^2$ answer: $21 - 20i$

Complex Solutions of a Quadratic Equation

Solve $x^2 + 9 = 0$

$$x^2 = -9$$

$$x = \pm\sqrt{-9}$$

$$x = \pm 3i$$

Solve $2x^2 - 5x + 6 = 0$ by the quadratic formula.

$a = 2 \qquad b = -5 \qquad c = 6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 4(2)(6)}}{2(2)} = \frac{5 \pm \sqrt{25 - 48}}{4}$$

$$= \frac{5 \pm \sqrt{-23}}{4} = \frac{5 \pm \sqrt{23}i}{4} = \frac{5}{4} + \frac{\sqrt{23}}{4}i$$

Examples:

(a) Solve $x^2 + 7 = 0$

$$x = \pm\sqrt{7}\, i$$

(b) Solve $x^2 + 6x + 15 = 0$ by completing the square.

$$x^2 + 6x + \underline{9} = -15 + \underline{9}$$

$$(x+3)^2 = -6$$

$$x + 3 = \pm\sqrt{-6}$$

$$x + 3 = \pm\sqrt{6}\,i$$

$$x = -3 \pm\sqrt{6}\,i$$