

Zeros of Polynomial Functions

We already know that an n th-degree polynomial can have at most n real zeros. Now that we know about complex numbers, we can say that every n th-degree polynomial function has exactly n zeros.

The Fundamental Theorem of Algebra:

If $f(x)$ is a polynomial function of degree n , where $n > 0$, then f has at least one zero in the complex number system.

From this we get:

The Linear Factorization Theorem:

If $f(x)$ is a polynomial function of degree n , where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \dots (x - c_n)$$

where c_1, c_2, \dots, c_n are complex numbers.

*Note: c_1, c_2, \dots, c_n may not be distinct. Also remember that every real number is also a complex number, so some of the numbers c_1, c_2, \dots, c_n may be real numbers.

*Because these theorems tell us that the zeros and factors exist, they are called existence theorems. They don't tell us how to find the zeros and factors, just that they exist.

Examples:

$f(x) = x - 4$ has exactly one zero: $x = 4$

$f(x) = x^2 - 10x + 25 = (x-5)(x-5)$ had exactly 2 zeros: $x=5, 5$

$f(x) = x^3 + 9x = x(x^2 + 9) = x(x-3i)(x+3i)$ has exactly 3 zeros:
 $0, 3i, -3i$

$f(x) = x^4 - 1 = (x+1)(x-1)(x+i)(x-i)$ has exactly 4 zeros: $\pm 1, \pm i$

If the coefficients of a polynomial function are integers, then the leading coefficient and constant term give us clues about the zeros.

The Rational Zero Test:

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 + a_0$ has non-zero integer coefficients, and $a_n \neq 0$, $a_0 \neq 0$, then any rational zero of f will be of the form p/q , where p is a factor of a_0 and q is a factor of a_n .

**This will help us decide what values to try when looking for the zeros of a function.

Example: Look at $f(x) = 2x^3 + 6x + 5$.

The Rational Zero test tells us that our zeros will be from the following list:

Rational zeros: $\frac{\text{factors of } 5}{\text{factors of } 2} = \frac{\pm 1}{1}, \frac{\pm 5}{1}, \frac{\pm 1}{2}, \frac{\pm 5}{2} \left(\pm 1, \pm 5, \frac{\pm 1}{2}, \frac{\pm 5}{2} \right)$

Example: List the possible rational zeros of the polynomial function $f(x) = 3x^5 + x^4 + 4x^3 - 2x^2 + 8x - 5$.

$$\frac{\text{factors of } 5}{\text{factors of } 3} = \pm 1, \pm 5, \frac{\pm 1}{3}, \frac{\pm 5}{3}$$

Example: Find the zeros of $f(x) = x^3 - 7x - 6$.

1. We need to solve $x^3 - 7x - 6 = 0$.
2. Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$
3. Use synthetic division until you find a zero.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

This means $f(x) = (x+1)(x^2 - x - 6)$

4. Continue factoring or doing synthetic division to find the other factors.

$$f(x) = (x+1)(x-3)(x-2)$$

5. The zeros are $-1, 3, 2$.

Example: Find all rational zeros of $f(x) = x^3 - 20x^2 + 23x + 10$.

$$p: \pm 1, \pm 2, \pm 5, \pm 10$$

$$q: \pm 1, \pm 3$$

$$\frac{p}{q}: \pm 1, \pm 2, \pm 5, \pm 10, \frac{\pm 1}{3}, \frac{\pm 2}{3}, \frac{\pm 5}{3}, \frac{\pm 10}{3}$$

$$f(x) = (x-2)(3x+1)(x-5)$$

$$\text{zeros are } 2, -\frac{1}{3}, 5$$

Conjugate Pairs

Look at $f(x) = x^4 - x^3 + 7x^2 - 9x - 18$.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

$$\begin{array}{r|rrrrr}
 -1 & 1 & -1 & 7 & -9 & -18 \\
 & & -1 & 2 & -9 & 18 \\
 \hline
 & 1 & -2 & 9 & -18 & 0
 \end{array}$$

Now $f(x) = (x+1)(x^3 - 2x^2 + 9x - 18)$. Repeat the process.

$$\begin{array}{r|rrrr}
 2 & 1 & -2 & 9 & -18 \\
 & & 2 & 0 & 18 \\
 \hline
 & 1 & 0 & 9 & 0
 \end{array}$$

Now $f(x) = (x+1)(x-2)(x^2+9)$.

Solve: $(x+1)(x-2)(x^2+9) = 0$
 $x+1=0$ or $x-2=0$ or $x^2+9=0$
 $x = -1$ or $x = 2$ or $x^2 = -9$
 $x = \pm 3i$

So, we have $f(x) = (x+1)(x-2)(x+3i)(x-3i)$

*Note: When we have a factor like $x+3i$, we must also have its conjugate $x-3i$ as a factor.

Multiply these conjugates:

$$(3 + \sqrt{2})(3 - \sqrt{2}) = 9 - 2 = 7$$

$$(4 + 3i)(4 - 3i) = 16 - 9i^2 = 16 + 9 = 25$$

$$(2 - i)(2 + i) = 4 - i^2 = 4 + 1 = 5$$

$$(x + 2i)(x - 2i) = x^2 - 4i^2 = x^2 + 4$$

$$(x + 7i)(x - 7i) = x^2 - 49i^2 = x^2 + 49$$

Factor:

$$x^2 + 25 = (x+5i)(x-5i)$$

$$x^2 + 36 = (x+6i)(x-6i)$$

$$x^2 + 12 = (x+2\sqrt{3}i)(x-2\sqrt{3}i)$$

$$x^2 + 5 = (x+\sqrt{5}i)(x-\sqrt{5}i)$$

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has real coefficients. If $a + bi$ (where $b \neq 0$) is a zero of the function, then we know that $a - bi$ is also a zero of the function.

Example: Find a 4th degree polynomial that has 2, -1, and $3i$ as zeros.

$$f(x) = (x-2)(x+1)(x-3i)(x+3i)$$

$$f(x) = (x^2 - x - 2)(x^2 + 9)$$

$$f(x) = x^4 - x^3 + 7x^2 - 9x - 18$$

Factoring a Polynomial

The Linear Factorization Theorem shows we can write any n th-degree polynomial as the product of n linear terms. But some of those terms may be complex. If we don't want to worry about complex factors, we can still write the polynomial as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Example: $f(x) = x^4 - x^3 + 7x^2 - 9x - 18$ can be factored to
 $f(x) = (x-2)(x+1)(x^2 + 9)$.

**We say that $x^2 + 9$ is irreducible over the reals.

$$\begin{aligned} 3. \text{ So, } f(x) &= (x^2 - x - 6)(x^2 - 2x + 10) \\ &= (x-3)(x+2)(x^2 - 2x + 10) \end{aligned}$$

4. The zeros are 3, -2, $1+3i$, $1-3i$

Example: Write the polynomial function $f(x) = x^4 + 5x^2 - 36$ as the product of linear factors, and list all zeros.

$$\begin{aligned} f(x) &= (x^2 + 9)(x^2 - 4) \\ f(x) &= (x + 3i)(x - 3i)(x+2)(x-2) \\ \text{The zeros are } &\pm 3i, \pm 2 \end{aligned}$$

****Note:** Real zeros are the only zeros that appear as x-intercepts on a graph. A polynomial function's complex zeros must be found algebraically.

Other Tests for Zeros of Polynomials

Descarte's Rule of Signs:

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of **positive real zeros** of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of **negative real zeros** of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

Two notes about Descartes's Rule of Signs:

1. A variation in sign means that two consecutive coefficients have opposite signs.
2. When we count the zeros, we must count their multiplicities.

Example: Look at $f(x) = 7x^3 + 3x^2 - 5x + 9$.

$f(x)$ has two variations in sign; therefore, there are either two or no positive real zeros.

$$f(-x) = -7x^3 + 3x^2 + 5x + 9.$$

$f(-x)$ has one variation in sign; therefore, there is exactly one negative real zero.

Example: Find the number of variations in sign in $f(x) = 2x^6 + 3x^5 - x^4 - 9x^3 + x^2 + 5x - 7$ as well as the number of variations of sign in $f(-x)$. Then discuss the possible numbers of positive real zeros and the possible number of negative real zeros of this function.

- *3 changes in sign for $f(x)$*
- $f(-x) = 2x^6 - 3x^5 - x^4 + 9x^3 + x^2 - 5x - 7$, so there are 3 changes in sign for $f(-x)$
- *$f(x)$ has either 3 or 1 positive real zeros, and either 3 or 1 negative real zeros.*

The other test we will use when we have a very large list of possible rational zeros is the Upper and Lower Bounds Rule.

Definition: A real number b is an upper bound for the real zeros of f if there are no zeros of f greater than b .

A real number b is a lower bound for the real zeros of f if there are no zeros of f less than b .

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial function with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$ using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an upper bound for the real zeros of f .
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as either positive or negative), then c is a lower bound for the real zeros of f .

Example: Look at $f(x) = 3x^3 - 2x^2 + 15x - 10$.

1. Possible rational zeros: $\pm 1, \pm 2, \pm 5, \pm 10, \frac{\pm 1}{3}, \frac{\pm 2}{3}, \frac{\pm 5}{3}, \frac{\pm 10}{3}$
2. Using Descartes's Rule of Signs we know that there are 3 or 1 positive zeros because there are 3 sign changes.

$f(-x) = -3x^3 - 2x^2 - 15x - 10$ so we know there are no negative zeros since there are no sign changes.

3. Try one of the positive numbers, like $x=2$, using synthetic division.

$$\begin{array}{r|rrrr}
 2 & 3 & -2 & 15 & -10 \\
 & & 6 & 8 & 46 \\
 \hline
 & 3 & 4 & 23 & 32
 \end{array}$$

This tells us that $x=2$ is not a zero, but since the last row of the synthetic division is all positive, we know that $x=2$ is an upper bound for zeros and we don't have to even try 5, 10, or $\frac{10}{3}$.

4. We only need to try 1 , $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{5}{3}$ to find zeros.

$$\begin{array}{r|rrrr}
 \frac{2}{3} & 3 & -2 & 15 & -10 \\
 & & 2 & 0 & 10 \\
 \hline
 & 3 & 0 & 15 & 0
 \end{array}$$

5. So, $f(x) = (x - \frac{2}{3})(3x^2 + 15)$.

$x = \frac{2}{3}$ is the only real zero.

From $(3x^2 + 15)$ we get complex zeros $x = \pm\sqrt{5}i$

****Descarte's Rule of Signs and the Upper and Lower Bounds Test are only tools to help narrow down some of the possibilities when finding zeros.**