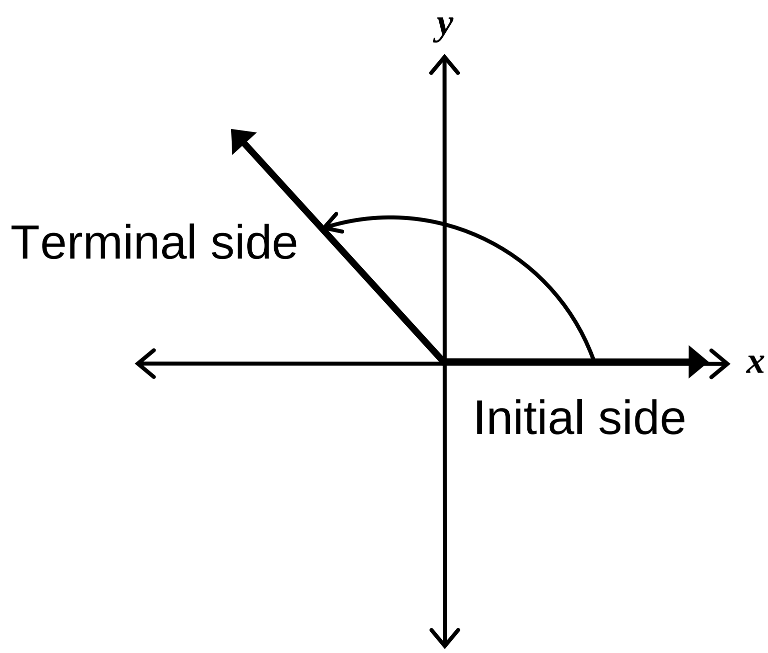
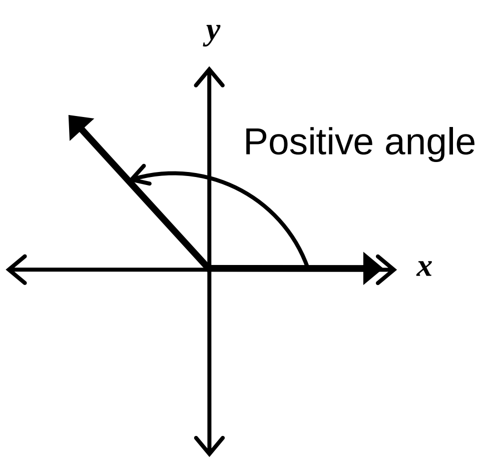
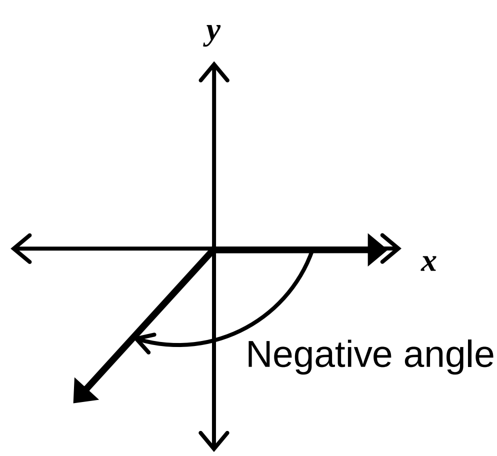
The Unit Circle

- An angle is in standard position when the angle's vertex is at the origin of a coordinate system and its initial side coincides with the positive *x*-axis.



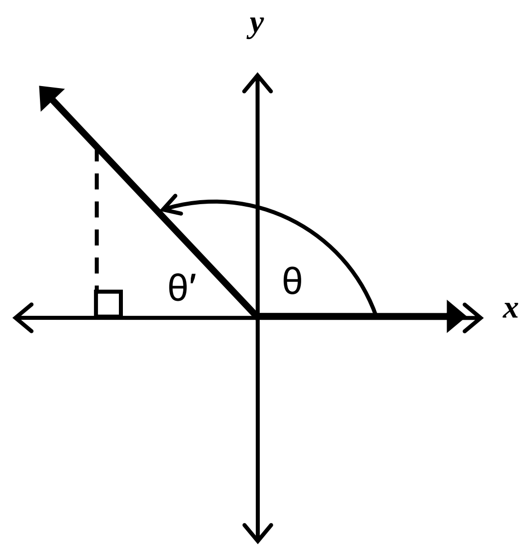
- A positive angle is generated by a counterclockwise rotation; whereas a negative angle is generated by a clockwise rotation.

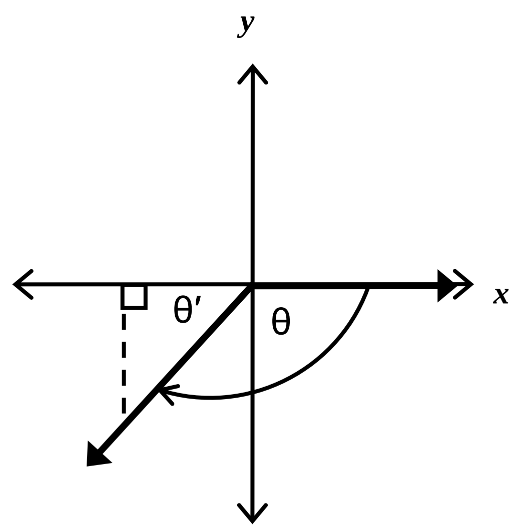


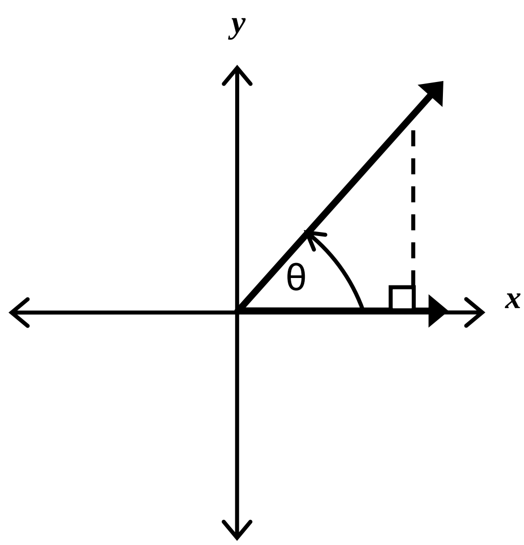


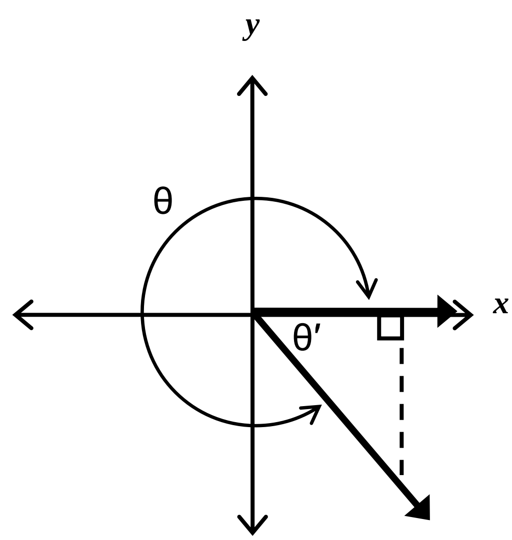
- The <u>reference angle</u> for an angle in standard position is the acute angle that the terminal side makes with the x-axis.
- The <u>reference triangle</u> is the right triangle formed which includes the reference angle.

The reference angle is θ′.









Example: Find the reference angle for the following angles.

a) $\theta = 125°$ answer: $\theta' = 180° - 125° = 55°$

b) $\theta = 5$ (radians) answer: since 5 radians ≈ 286° (Q4)
$\theta' = 2\pi - 5 \approx 1.2832$ or
$\theta' = 360° - 286° = 74°$

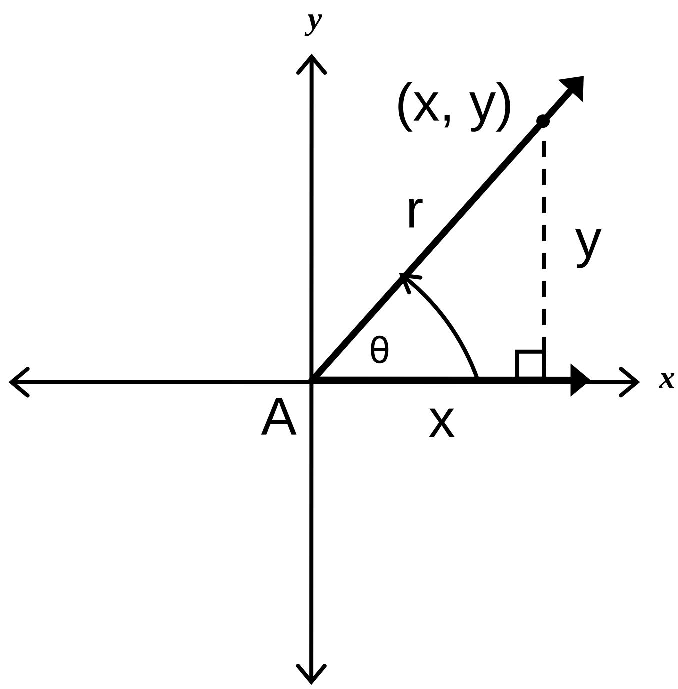
c) $\theta = 210°$ answer: Q3, so $180° + \theta' = 210°$
so $210° - 180° = 30°$

d) $\theta = 4.1$ (radians) answer: since $4.1 \approx 234.9°$ (Q3)
$\theta' = 4.1 - \pi \approx .9584$ or
$\theta' = 234.9° - 180° = 54.9°$

e) $\theta = \frac{-5\pi}{4}$ answer: $\frac{\pi}{4}$

f) $\theta = -100°$ answer: 80°

Definition of Trig Values for Acute Angles



$$\sin A = \frac{opp}{hyp} = \frac{y}{r} \qquad \csc A = \frac{r}{y}$$

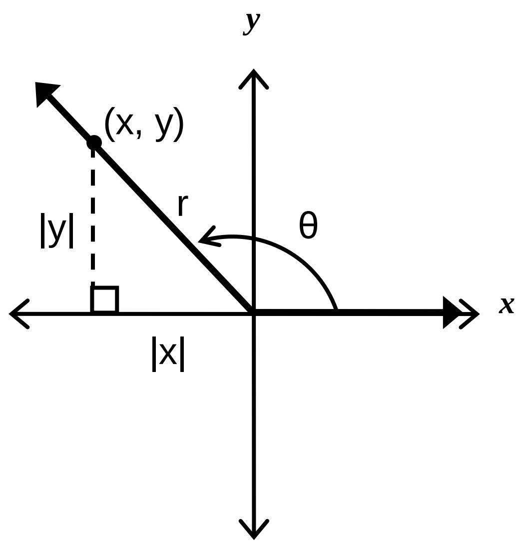
$$\cos A = \frac{adj}{hyp} = \frac{x}{r} \qquad \sec A = \frac{r}{x}$$

$$\tan A = \frac{opp}{adj} = \frac{y}{x} \qquad \cot A = \frac{x}{y}$$

with x, y, and $r \neq 0$.

Definition of Trig Values of Any Angle

Let θ be and angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$. Then

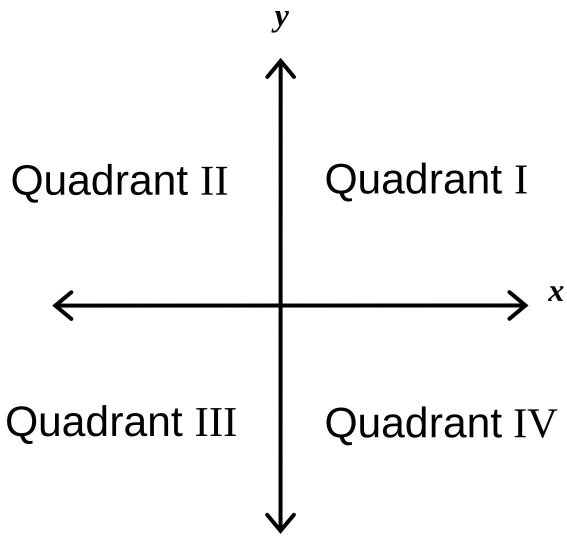


$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}, \quad y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}, \quad x \neq 0$$

$$\tan\theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

*<u>**Note**</u>: The value of r is always positive, but the signs on x and y depend on the point (x, y), which will change depending on which quadrant (x, y) is in.



Quad I (+,+)

$$\sin A = \frac{y}{r} = \frac{+}{+} = +$$
$$\cos A = \frac{x}{r} = \frac{+}{+} = +$$
$$\tan A = \frac{y}{x} = \frac{+}{+} = +$$
$$\csc A = \frac{r}{y} = \frac{+}{+} = +$$
$$\sec A = \frac{r}{x} = \frac{+}{+} = +$$
$$\cot A = \frac{x}{y} = \frac{+}{+} = +$$

Quad II (-,+)

$$\sin A = \frac{y}{r} = \frac{+}{+} = +$$
$$\cos A = \frac{x}{r} = \frac{-}{+} = -$$
$$\tan A = \frac{y}{x} = \frac{+}{-} = -$$
$$\csc A = \frac{r}{y} = \frac{+}{+} = +$$
$$\sec A = \frac{r}{x} = \frac{+}{-} = -$$
$$\cot A = \frac{x}{y} = \frac{-}{+} = -$$

Quad III (-,-)

$$\sin A = \frac{y}{r} = \frac{-}{+} = -$$
$$\cos A = \frac{x}{r} = \frac{-}{+} = -$$
$$\tan A = \frac{y}{x} = \frac{-}{-} = +$$
$$\csc A = \frac{r}{y} = \frac{+}{-} = -$$
$$\sec A = \frac{r}{x} = \frac{+}{-} = -$$
$$\cot A = \frac{x}{y} = \frac{-}{-} = +$$

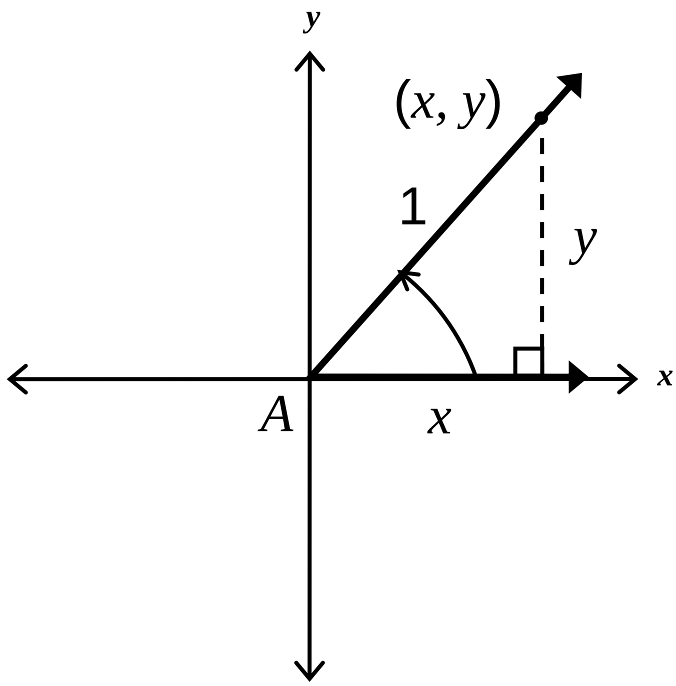
Quad IV (+,-)

$$\sin A = \frac{y}{r} = \frac{-}{+} = -$$
$$\cos A = \frac{x}{r} = \frac{+}{+} = +$$
$$\tan A = \frac{y}{x} = \frac{-}{+} = -$$
$$\csc A = \frac{r}{y} = \frac{+}{-} = -$$
$$\sec A = \frac{r}{x} = \frac{+}{+} = +$$
$$\cot A = \frac{x}{y} = \frac{+}{-} = -$$

allsintancos!!!!

(all)(sin)(tan)(cos) → What functions are positive, starting with Quadrant I.

Let's look at what the reference triangles look like when we choose (x, y) so that r=1.



Now we have:

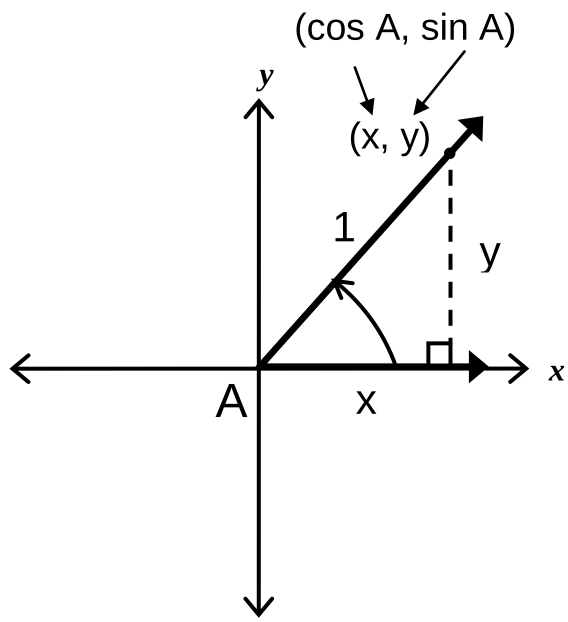
$$\sin A = \frac{y}{r} = \frac{y}{1} = y \qquad \csc A = \frac{1}{y}$$

$$\cos A = \frac{x}{r} = \frac{x}{1} = x \qquad \sec A = \frac{1}{x}$$

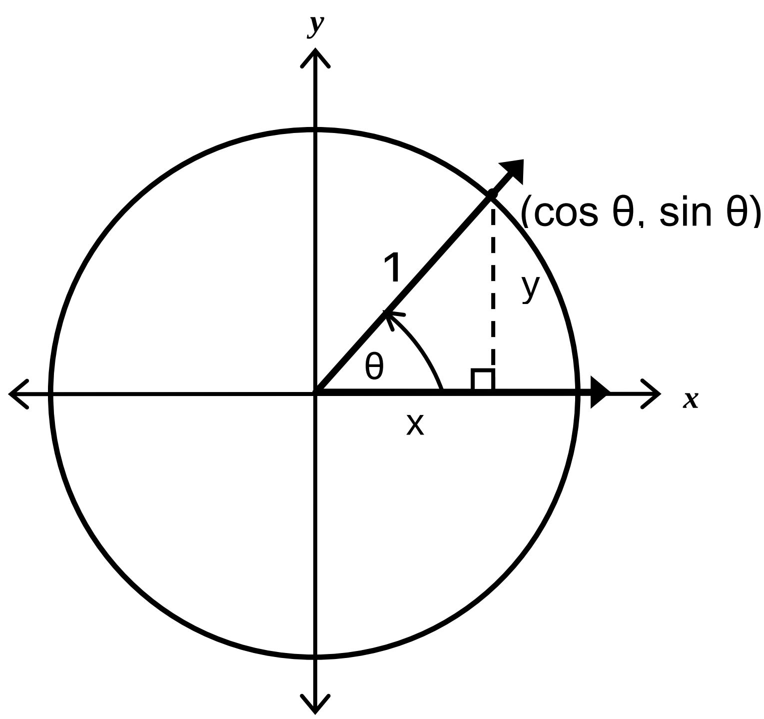
$$\tan A = \frac{y}{x} \qquad \cot A = \frac{x}{y}$$

**As long as r = 1, we have: cos A = x
sin A = y

The point (x,y) can be relaced with (cos A, sin A) because x=cos A and y=sin A.



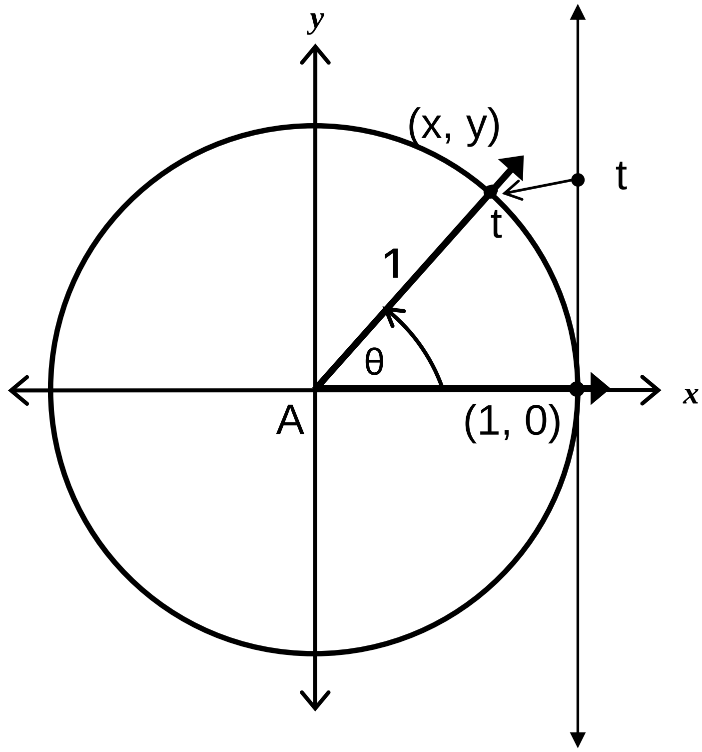
Look at a circle with radius = 1.



**For any coordinates on the unit circle, its coordinates are (cos θ, sin θ) where θ is an angle in standard position.

**If θ is <u>not</u> an acute angle, then we find the coordinates (x,y) (ie. cos θ, sin θ) by using the reference triangle.

Consider the unit circle.



Think of a number line wrapped around the circle. The length *t* maps to the point (x, y). We also know that

$$\theta = \frac{s}{r}$$

On our unit circle, s corresponds to the length of *t*, and *r* = 1, since the radius of the circle was chosen to be 1. This gives

$$\theta = \frac{t}{1} = t$$

This means that the length of *t* (in linear units) = the radian measure of θ.

By definition,

$$\sin t = y \qquad \csc t = \frac{1}{y}, \quad y \neq 0$$

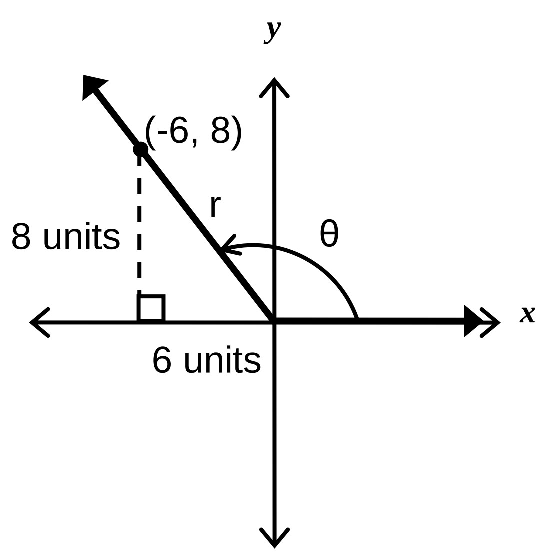
$$\cos t = x \qquad \sec t = \frac{1}{x}, \quad x \neq 0$$

$$\tan t = \frac{y}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

<u>Note</u>: This is just an alternative way of looking at angles on the unit circle. Since t (in linear units) = θ (in radians), then we can also substitute θ in the above definition.

<u>Evaluating Trig Functions</u>

Let (-6, 8) be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.



Solution:

First, find r.

Think of positive lengths for legs of reference triangle:

$$r^2 = |x|^2 + |y|^2$$
$$r^2 = 6^2 + 8^2$$
$$r^2 = 100$$
$$r = \sqrt{100}$$
$$r = 10$$

<u>or</u>

Use point (-5, 6) directly into the definition:

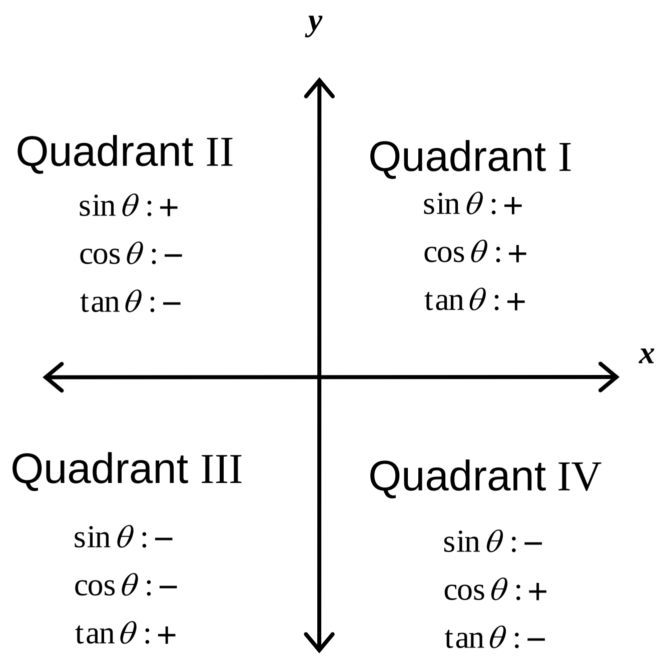
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-6)^2 + (8)^2}$$
$$r = \sqrt{36 + 64}$$
$$r = \sqrt{100}$$
$$r = 10$$

$$\sin\theta = \frac{y}{r} = \frac{8}{10} = \frac{4}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-6}{10} = \frac{-3}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{8}{-6} = \frac{-4}{3}$$

Consider again <u>allsintancos.</u>



We often need to use this to find trig values.

Example: Let θ be an angle in the third quadrant such that cos θ = -1/4. Find sin θ and tan θ.

a) Find sin θ. Use the Pythagorean identity.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(\frac{-1}{4}\right)^2 = 1$$

$$\sin^2\theta + \frac{1}{16} = 1$$

$$\sin^2\theta = \frac{15}{16}$$

$$\sin\theta = \pm\frac{\sqrt{15}}{4}$$

Since θ is in the 3rd quadrant, and sine is negative there, we know that

$$\sin\theta = \frac{-\sqrt{15}}{4}$$

b) Find tan θ. Use the quotient identity for tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{-\sqrt{15}}{4}}{\frac{-1}{4}} = \frac{-\sqrt{15}}{4}\cdot\frac{4}{-1} = \sqrt{15}$$

Example: Let cos θ = 8/17 and tan θ < 0. Find sin θ.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(\frac{8}{17}\right)^2 = 1$$

$$\sin^2\theta + \frac{64}{289} = 1$$

$$\sin^2\theta = \frac{225}{289}$$

$$\sin\theta = \pm\frac{15}{17}$$

Since tan θ < 0, we know that θ must be in either the 2nd or 4th quadrant. Since we have a positive cos θ, our angle must be in the 4th quadrant. Thus, we must have

$$\sin\theta = -\frac{15}{17}.$$

Example: Let csc θ = 4 and cos θ < 0. Find sec θ.

Since cosecant and sine are reciprocals, we know sin θ=¼.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = \frac{15}{16}$$

$$\cos\theta = \frac{\pm\sqrt{15}}{4}$$

Since cos θ < 0, we know that θ must be in either the 2nd or 3rd quadrant. Since we have a positive csc θ, our angle must be in the 2nd quadrant. Thus,

$$\cos\theta = \frac{-\sqrt{15}}{4}$$

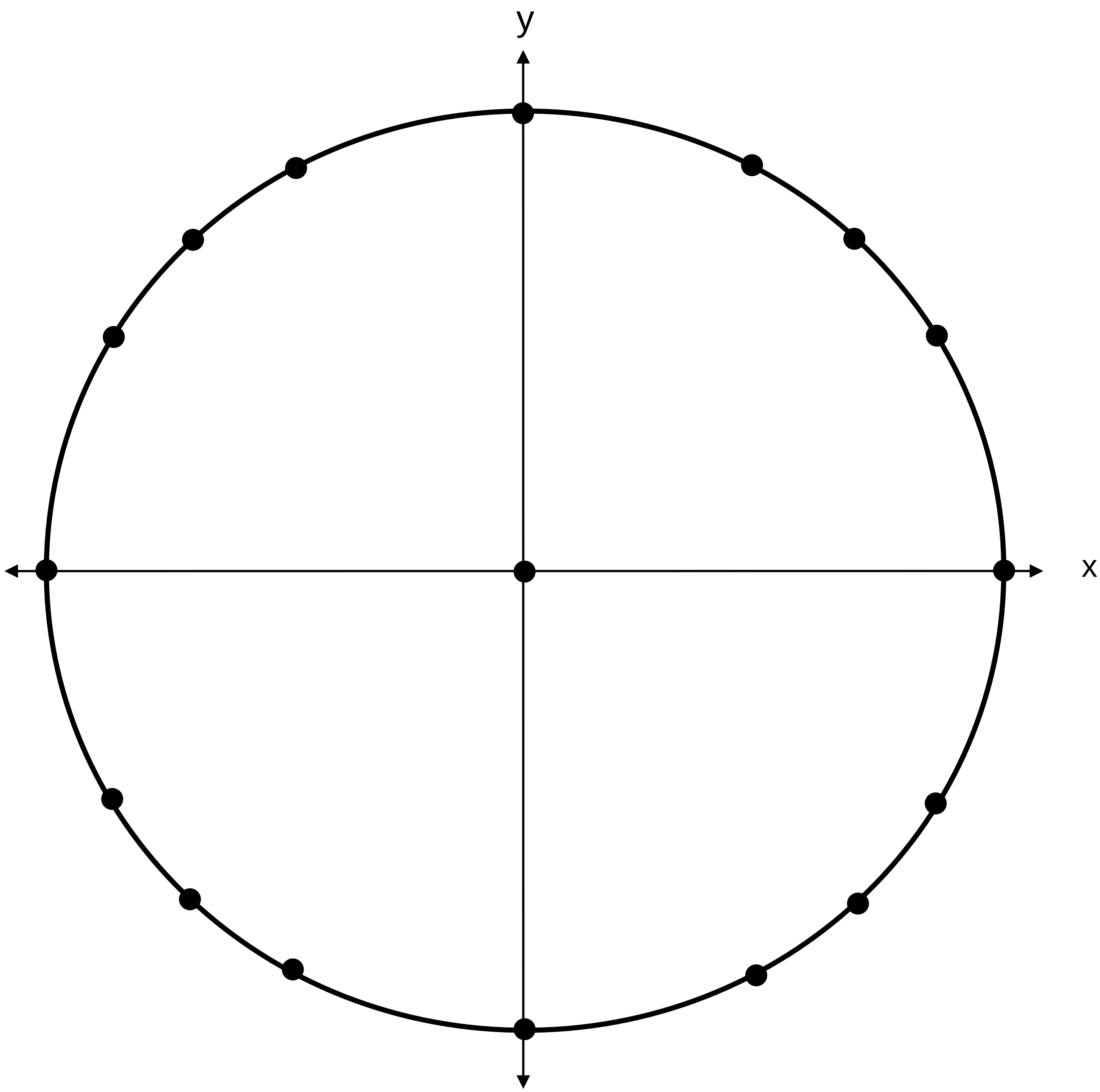
Since $\sec\theta = \frac{1}{\cos\theta}$ we have

$$\sec\theta = \frac{4}{-\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{-4\sqrt{15}}{15}$$

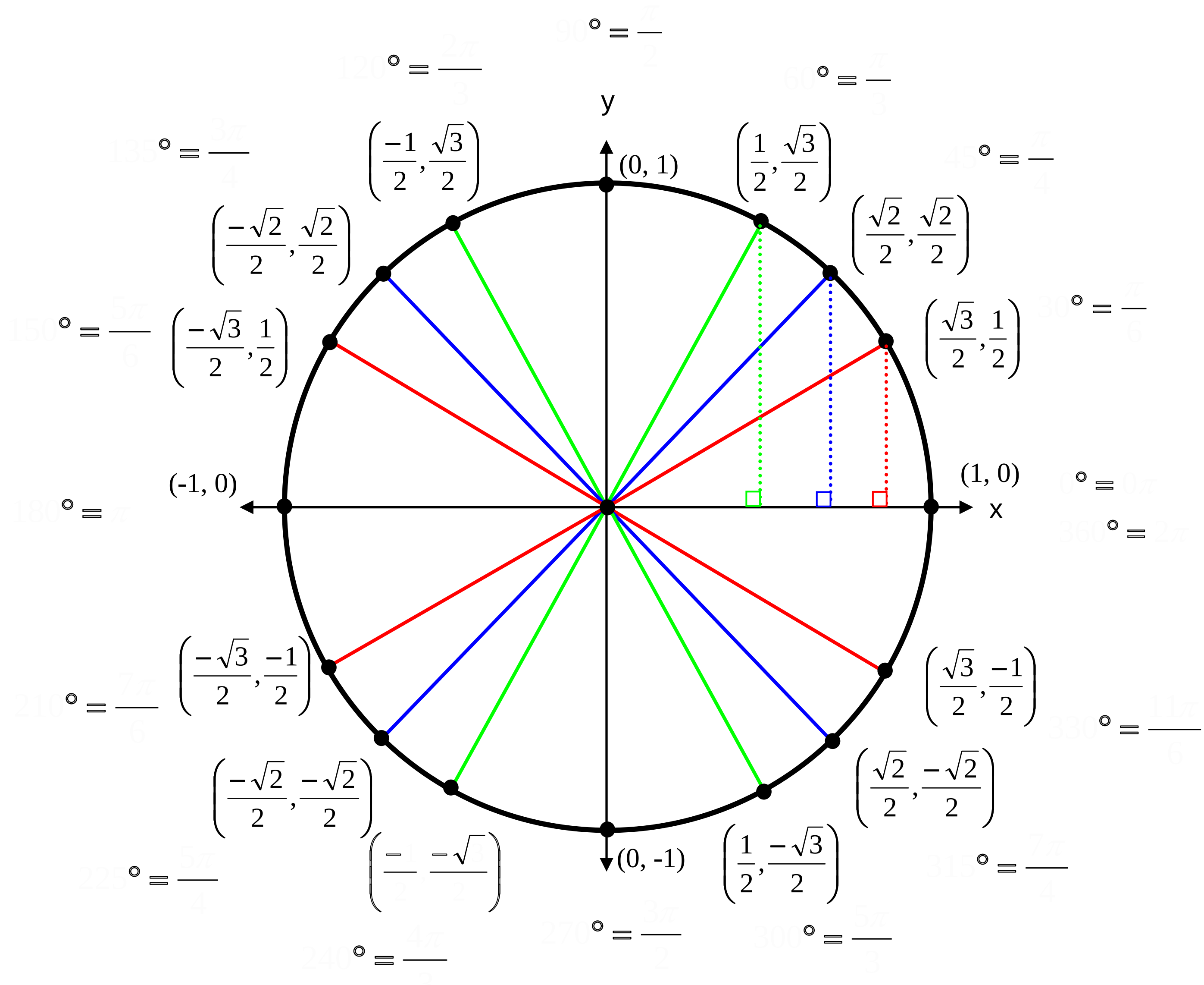
<u>The Unit Circle</u>

Because we so often use the special angles of 45°, 30°, 60° (π/4, π/3, π/6), it is helpful to memorize the angles and coordinates that correspond to these angles around the unit circle.

The Unit Circle



The Unit Circle



The cosine of angle θ is the x-coordinate.
The sine of angle θ is the y-coordinate.

$(x,y) = (\cos \theta, \sin \theta)$

Example: Using the unit circle, find the following.

a) $\sin\frac{3\pi}{4}$ answer: $\frac{\sqrt{2}}{2}$

b) $\cos\frac{-\pi}{2}$ answer: 0

c) $\tan 300°$ answer: $\frac{-\sqrt{3}}{2}\cdot\frac{2}{1} = -\sqrt{3}$

d) $\csc\pi$ answer: $\frac{1}{0} = \textit{undefined}$

<u>Even and Odd Trig Functions</u>

Look at $\frac{\pi}{3}$ and $\frac{-\pi}{3}$.

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \text{and } \cos\frac{-\pi}{3} = \frac{1}{2}$$

Because cos (θ) = cos (-θ), we say that cosine is an <u>even</u> function.

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{and } \sin\frac{-\pi}{3} = \frac{-\sqrt{3}}{2}$$

This is the *<u>opposite</u>* of $\sin\frac{\pi}{3}$

Because sin(-θ) = -sin(θ), we say that sine is an <u>odd</u> function.

Even and Odd Trigonometric Functions

The cosine and secant functions are ***even***.

$$\cos(-\theta) = \cos(\theta) \qquad \sec(-\theta) = \sec(\theta)$$

The sine, cosecant, tangent, and cotangent functions are ***odd***.

$$\sin(-\theta) = -\sin(\theta) \qquad \csc(-\theta) = -\csc(\theta)$$
$$\tan(-\theta) = -\tan(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

Periodic Functions

Remember that coterminal angles have the same terminal side. Thus, they will have the same coordinates on the unit circle and the same trig values.

$\sin 20° = \sin(20°+360°) = \sin(20°+720°)$ etc.

$\cos\frac{\pi}{2} = \cos(\frac{\pi}{2} + 2\pi) = \cos(\frac{\pi}{2} + 4\pi)$ etc.

We say: $\sin\theta = \sin(\theta+2k\pi)$
$\cos\theta = \cos(\theta+2k\pi)$, where k is an integer.

When functions behave like this, we call them periodic functions.

Definition: A function f is periodic if there exists a positive real number c such that

$$f(x) = f(x + c)$$

for all x in the domain of f. The smallest number c, for which the function is periodic is called the period.

In our case we have: $\sin(\theta + 2k\pi) = \sin\theta$
$\cos(\theta + 2k\pi) = \cos\theta$

The smallest value that $2k\pi$ can be is if k = 1, which gives us $2(1)\pi = 2\pi$. Thus,

The period of $f(x) = \sin\theta$ is 2π.
The period of $f(x) = \cos\theta$ is 2π.

Example: Find the following:

a) $\sin\frac{25\pi}{4}$

$$\sin\frac{25\pi}{4} = \sin\left(\frac{\pi}{4} + \frac{24\pi}{4}\right) = \sin\left(\frac{\pi}{4} + 6\pi\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

b) cos 840°

cos1200° = cos(120°+1080°)= cos(120°+ 3(360°)) = cos(120°) = -½

c) $\cos\frac{3\pi}{3}$

$$\cos\frac{38\pi}{3}=\cos\left(12\frac{2}{3}\pi\right)=\cos\left(\frac{2\pi}{3}+12\pi\right)=\cos\frac{2\pi}{3}=\frac{-1}{2}$$

Domain and Range of Sine and Cosine

Let θ = any radian angle measure

Then $x = \cos\theta$ and $y = \sin\theta$

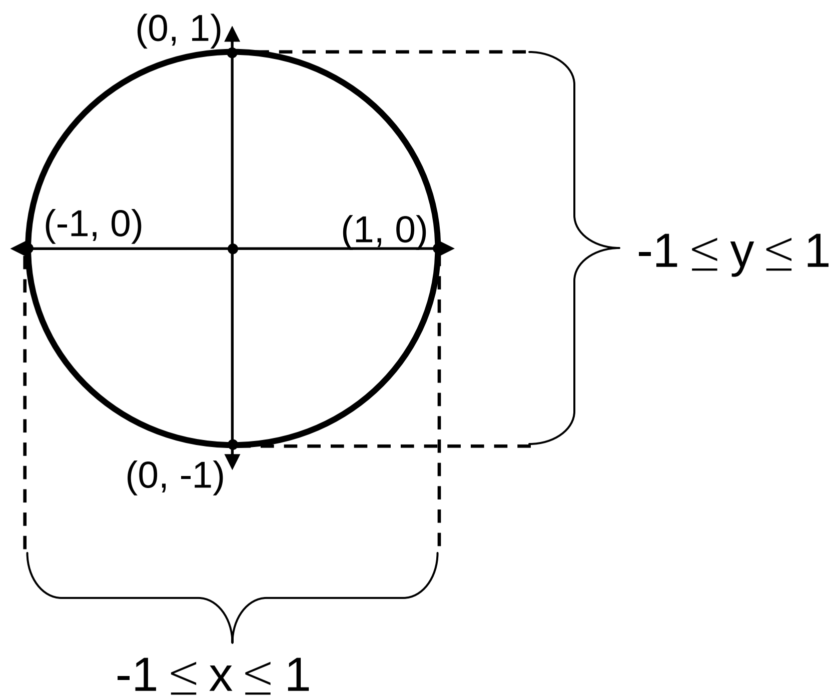
Domain: What values can you put for θ ?

You can put any real number in for θ (remember rotations around the circle).

Range: What values will you get for cos θ and sin θ?

You will always get a number from -1 to 1.

(On the unit circle, you only use the values from -1 to 1 for any coordinates on the unit circle, no matter how many rotations the angle has.)



<u>Using the Calculator</u>

Example: Find the following:

a) cot 400°

(Degree mode) [TAN] (400) [x^{-1}] [ENTER] ≈ 1.1918

b) cos -8

(Radian mode) [COS] (-8) [ENTER] ≈ -0.1455

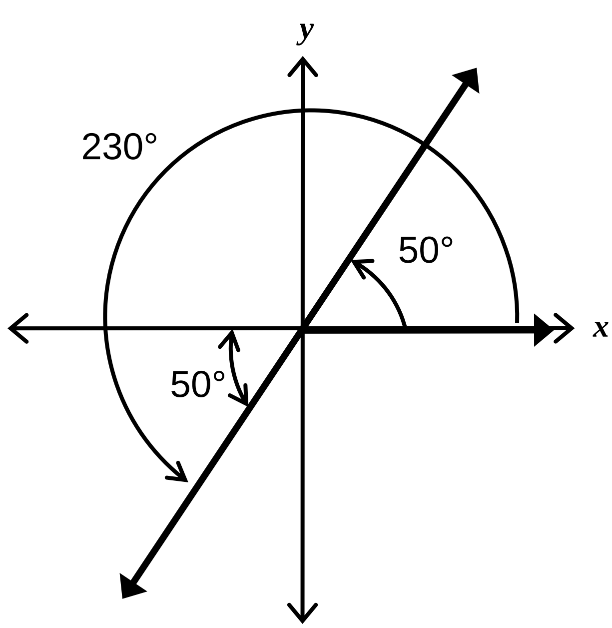
c) csc $\frac{5\pi}{7}$

(Radian mode) [SIN] (5π ÷ 7) [x^{-1}] [ENTER] ≈ -1.1547

Example: Use a calculator to solve tan θ = 1.192 for $0 \le \theta \le 2\pi$

(Radian mode) [2nd] [TAN] (3.715) [ENTER] ≈ .873 radians

(Degree mode) [2nd] [TAN] (3.715) [ENTER] ≈ 50°



Remember allsintancos. We need to consider all of the angles that have reference angles of 50° where the tangent is positive. This happens in the 1st and 3rd quadrants, so our answer is

θ = 50° or 230°