Law of Cosines

To solve an oblique triangle, we need to know at least one side and any two other parts of the triangle.

4 cases for oblique triangles:

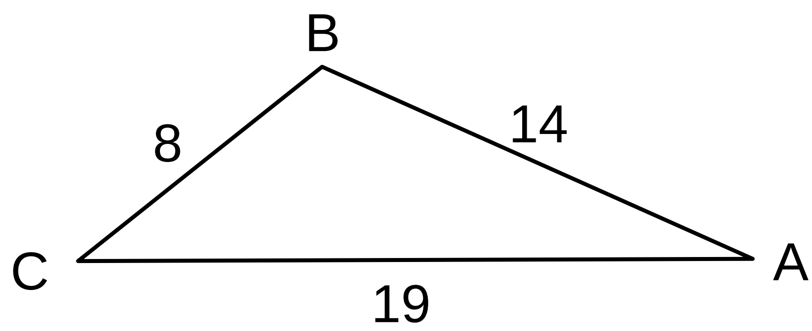
1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

We use the Law of Sines for the first two cases. For the last 2 cases we use the Law of Cosines.

Law of Cosines

Standard Form	Alternative Form
$a^2 = b^2 + c^2 - 2bc\cos A$	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac\cos B$	$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Example: Find the three angles of the triangle.



Find angle opposite the longest side first.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{8^2 + 14^2 - 19^2}{2(8)(14)}$$

$$\cos B \approx -0.45089$$

Since $\cos B$ is negative, we know by allsintancos that it is a 2nd quadrant angle, which means it is obtuse.

Using a calculator and $\cos^{-1}$, we find that m∠B≈116.8°

You can use either the Law of Cosines or the Law of Sines to find the next angle.

$$\frac{\sin A}{8} = \frac{\sin 116.8°}{19}$$

$$\sin A = \frac{8 \sin 116.8°}{19} \approx 0.37583$$

If it weren't for the fact that we already have an obtuse angle in our triangle, we would have to consider that there are 2 angles that have sine=0.37583 (one acute and one obtuse). The acute angle is 22.08° and the obtuse angle is 180°- 22.08° = 157.92°.

Therefore, for our triangle, m∠A=22.08°. Now find m∠C.

m∠C = 180°- 22.08°- 116.8 = 41.12°

***Note:** If the largest angle is obtuse, then the other 2 angles are acute. If the largest angle is acute, the other 2 angles will also be acute.

Example: Given *C* = 111°, *a* = 27, and *b* = 18, find the remaining side and two angles of the triangle.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 27^2 + 18^2 - 2(27)(18)\cos 111°$$

$$c^2 \approx 1404.33$$

$$c \approx 37.43$$

$$\frac{\sin A}{27} = \frac{\sin 111°}{37.43}$$

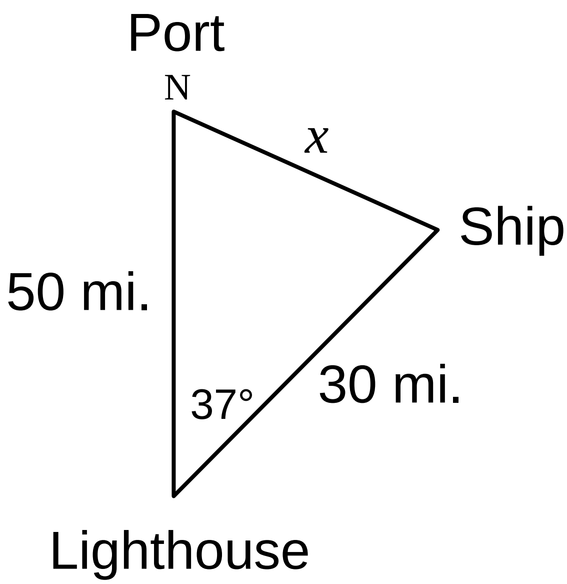
$$\sin A = \frac{27\sin 111°}{37.43}$$

$$A \approx 42.3°$$

$$B = 180° - 111° - 42.3° = 26.7°$$

Applications

Example: A port is 50 miles due north of a lighthouse. A ship is 30 miles from the lighthouse at a bearing of N 37° E. How far is the ship from the port?



$$x^2 = 50^2 + 30^2 - 2(50)(30)\cos 37°$$

$$x^2 \approx 1004.09$$

$$x \approx 31.69 \text{ miles}$$

Heron's Area Formula

Given any triangle with sides of lengths a, b, and c, the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$, the semi-perimeter of the triangle.

Example: Find the area of the triangular region with sides 50 feet, 58 feet, and 69 feet.

The semi-perimeter is $s = \frac{(50+58+69)}{2} = 88.5$

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

$$Area = \sqrt{88.5(88.5-50)(88.5-58)(88.5-69)}$$

$$Area \approx 1423.54 \ \ ft^2$$