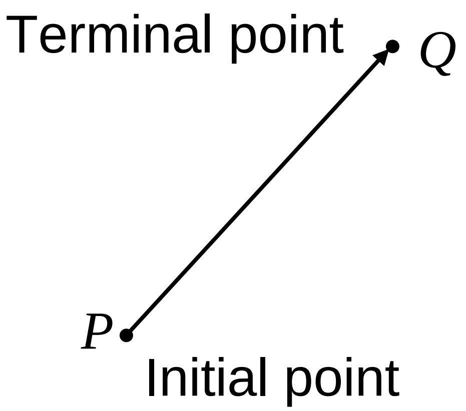
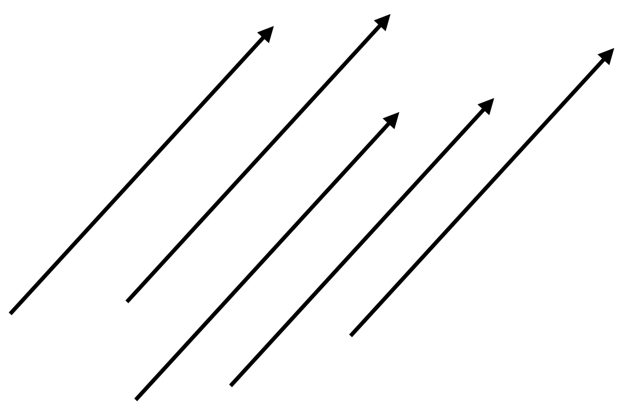
Vectors in the Plane

When quantities involve both magnitude and direction, we represent them using a directed line segment.



The magnitude (length) of the directed line segment is denoted by $\|PQ\|$ and can be found using the distance formula.

If directed line segments have the same magnitude and direction, then they are equivalent.



These are all equivalent to $\overrightarrow{PQ}$

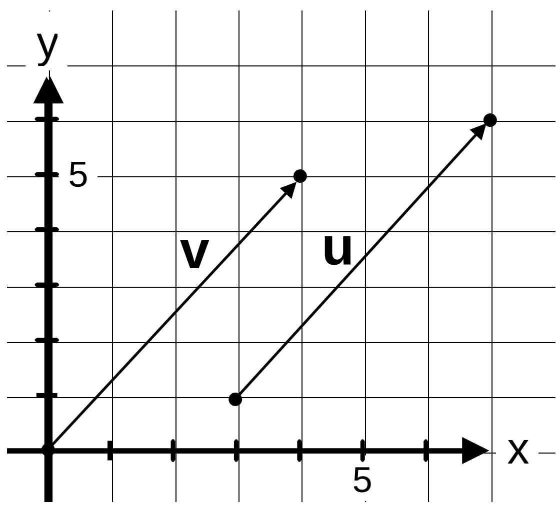
Note: Directed line segments are denoted with the half-arrow notation so as not to be confused with rays (which have direction, but no magnitude).

We represent the set of ALL equivalent directed line segments equivalent to $\overrightarrow{PQ}$ as a <u>vector **v** in the plane</u>. It is written

$$\mathbf{v} = \overrightarrow{PQ}$$

Vectors are denoted by lowercase, boldface letters such as **u**, **v**, and **w**.

Example: Let **v** be the vector from (0, 0) to (4, 5) and **u** be the vector from (3, 1) to (7, 6). Show that **v** = **u**.



To be equivalent, the vectors must have the same magnitude, direction, and slope.

1. Compare magnitudes using the distance formula.

$$\|v\| = \sqrt{(4-0)^2 + (5-0)^2} = \sqrt{16+25} = \sqrt{41}$$
$$\|u\| = \sqrt{(7-3)^2 + (6-1)^2} = \sqrt{16+25} = \sqrt{41}$$

2. Compare the directions.

Both are directed toward the upper right.

3. Find the slopes.

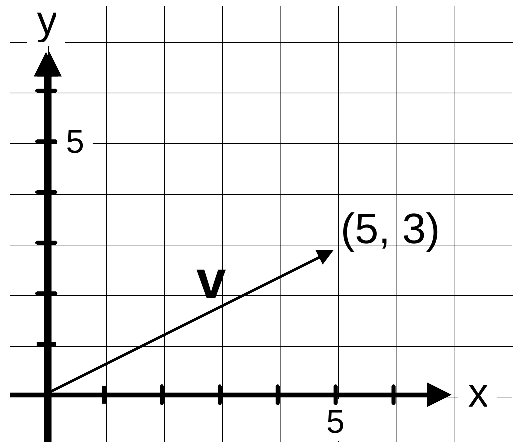
$$m_v = \frac{\Delta y}{\Delta x} = \frac{5-0}{4-0} = \frac{5}{4}$$

$$m_u = \frac{\Delta y}{\Delta x} = \frac{6-1}{7-3} = \frac{5}{4}$$

Since the magnitudes, slopes, and directions are the same, the vectors are equivalent.

<u>Component Form of a Vector</u>

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments. This vector is said to be in standard form.



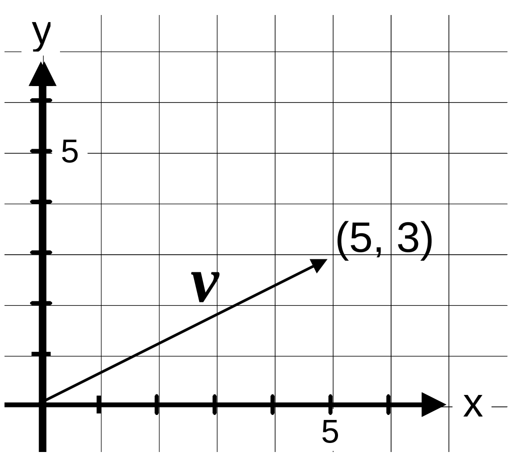
v is in standard form

A vector in standard form can be represented by the coordinates of its terminal point. This is the component form of a vector **v**, written

$$\boldsymbol{v} = \langle v_1, v_2 \rangle$$

where (v_1, v_2) is the terminal point.

Note: v_1 is the x-component (horizontal component)
v_2 is the y-component (vertical component)



$$\boldsymbol{v} = \langle 5, 3 \rangle$$

If both the initial point and the terminal point lie at the origin, it is called the zero vector and is denoted by

$$\boldsymbol{0} = \langle 0, 0 \rangle$$

Component Form of a Vector

The component form of the vector with initial point P = (p_1, p_2) and terminal point Q = (q_1, q_2) is

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \boldsymbol{v}$$

The <u>magnitude</u> (or length) of **v** is

$$\|\boldsymbol{v}\| = \sqrt{(q_1 - p_1)^2 + (q_{2^2} - p_2)^2} = \sqrt{v_1{}^2 + v_2{}^2}$$

If $\|v\| = 1$, **v** is the unit vector. Moreover, $\|v\| = 0$, if and only if **v** is the zero vector **0**.

<u>Definition</u>: Two vectors **u** = $\langle u_1, u_2 \rangle$, and **v** = $\langle v_1, v_2 \rangle$ are <u>equal</u> if and only if $u_1 = v_1$ and $u_2 = v_2$.

Example: Find the component form and the magnitude of the vector from (2, -4) to (5, 7).

Start with the terminal point when finding the component form.

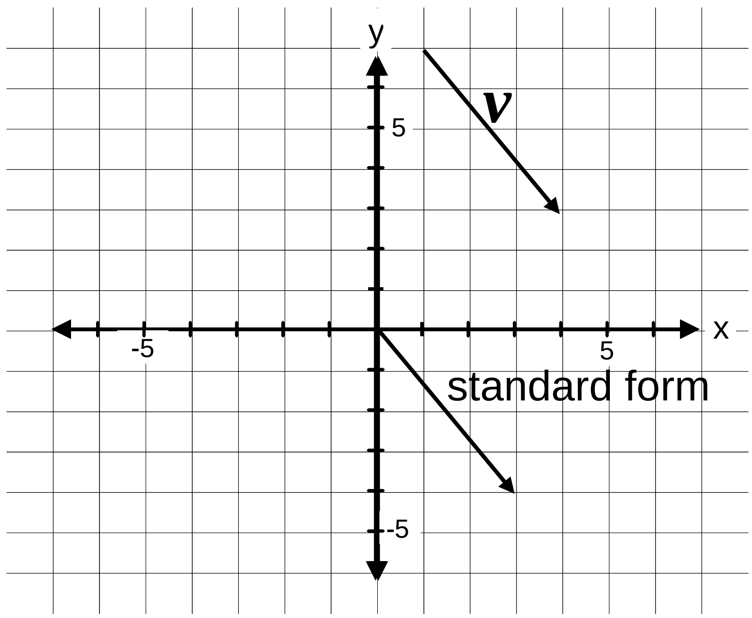
$$v = \langle 5 - 2, 7 - (-4) \rangle = \langle 3, 11 \rangle$$

$$\|v\| = \sqrt{(5-2)^2 + (7-(-4))^2} = \sqrt{9 + 121} = \sqrt{130}$$

Example: Find the component form and magnitude of the vector **v** that has (1, 7) as its initial point and (4, 3) as its terminal point.

$$v = \langle 4-1, 3-7 \rangle = \langle 3, -4 \rangle$$

$$\|v\| = \sqrt{(4-1)^2 + (3-7)^2} = \sqrt{9+16} = \sqrt{25} = 5$$



Vector Operations

2 basic vector operations

1. Scalar Multiplication
2. Vector Addition

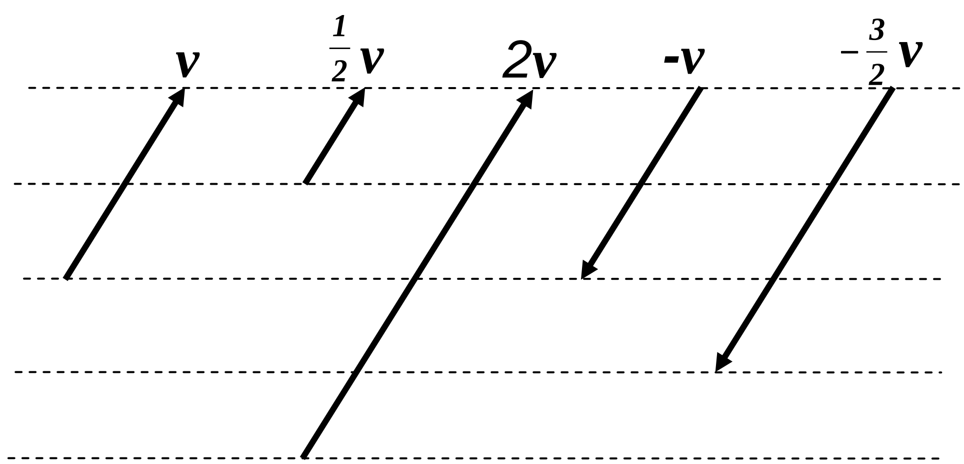
Scalar Multiplication

When multiplying a real number times a vector, we call the number a scalar.

The notation for scalar multiplication is $k\mathbf{u}$.

$$k\mathbf{u} \text{ is } |k| \text{ times as long as } \mathbf{u}.$$

When k is positive, then $k\mathbf{u}$ has the same direction as $\mathbf{u}$.
When k is negative, then $k\mathbf{u}$ has the opposite direction as $\mathbf{u}$.

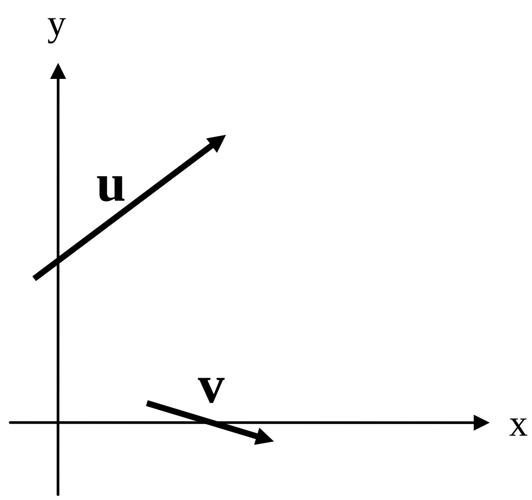


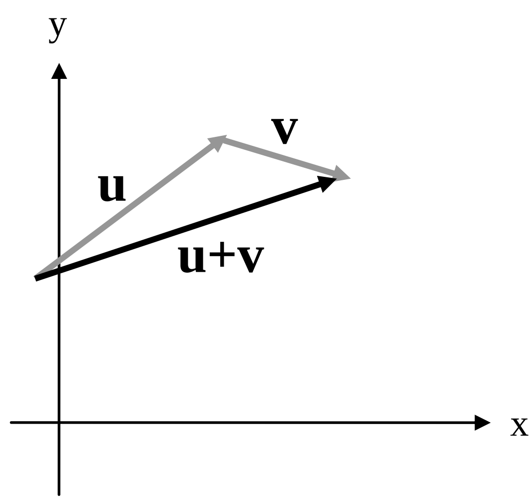
Example: Let $\mathbf{u} = \langle 3, -5 \rangle$. Find $6\mathbf{u}$.

$$6\mathbf{u} = 6\langle 3, -5 \rangle = \langle (6)(3), (6)(-5) \rangle = \langle 18, -30 \rangle$$

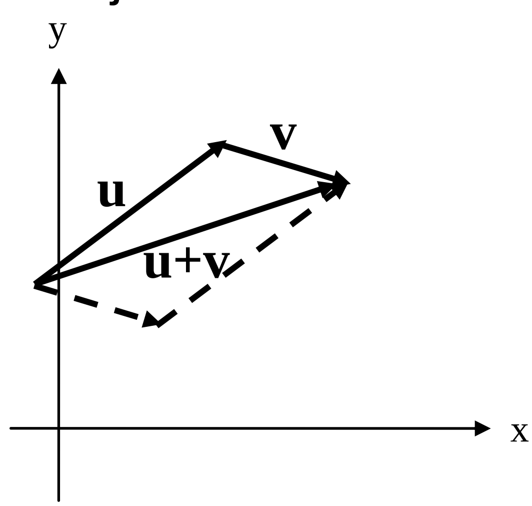
Vector Addition

To add 2 vectors geometrically, position them so that the initial point of one coincides with the terminal point of the other. The sum **u** + **v** is the vector formed joining the initial point of **u** to the terminal point of **v**.



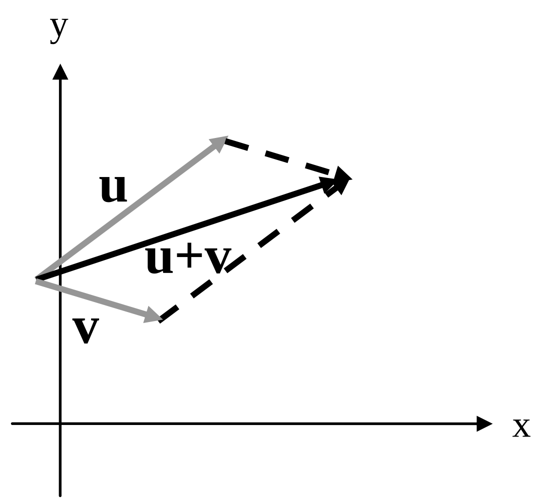


This is called the <u>parallelogram law</u> for vector addition because the vector **u** + **v**, is the diagonal of a parallelogram having **u** and **v** as its adjacent sides.

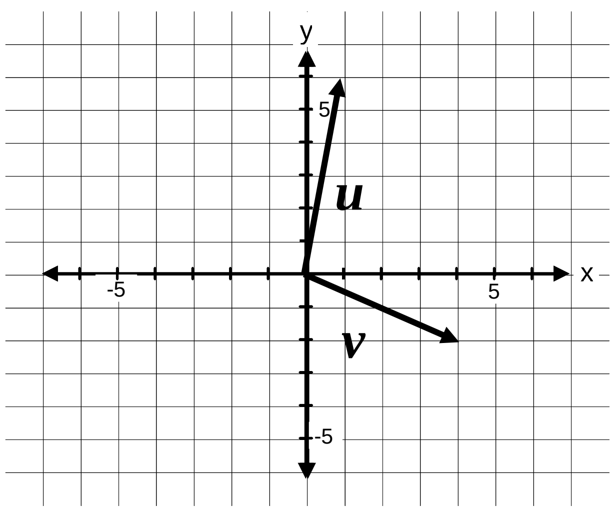


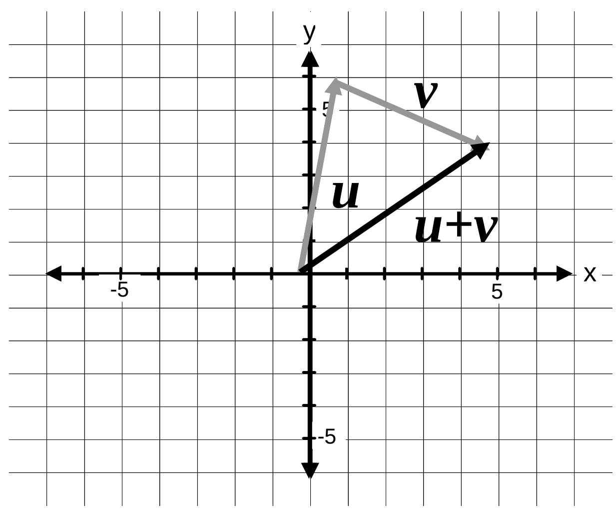
Alternate way of adding geometrically

You can also find the sum vector **u** + **v** by joining the initial points of both **u** and **v**. The sum vector is the vector that shares the same initial point and forms the diagonal of the parallelogram with sides **u** + **v.**



Example: Let **u** = $\langle 1, 6\rangle$ and **v** = $\langle 4, -2\rangle$. Find **u** + **v**.





Definitions of Vector Addition and Scalar Multiplication

Let **u** = $\langle u_1, u_2 \rangle$, and **v** = $\langle v_1, v_2 \rangle$, be vectors and let k be a scalar (a real number).

The sum of **u** and **v** is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1,\ u_2 + v_2 \rangle$$

The scalar multiple of k times **u** is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle, = \langle ku_1, ku_2 \rangle,$$

The negative of **v** is – **v** and

$$-\mathbf{v} = (-1)\,\mathbf{v} = \langle -v_1, -v_2 \rangle$$

The difference of **u** and **v** is the vector

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1,\ u_2 - v_2 \rangle$$

Example: Let **u** = $\langle -5, 2\rangle$ and **v** = $\langle 6, -3\rangle$. Find the following.

a) 4**u**

$$4\mathbf{u} = 4\langle -5, 2\rangle = \langle -20, 8\rangle$$

b) u + **v**

$$\mathbf{u} + \mathbf{v} = \langle -5 + 6,\ 2 + (-3)\rangle = \langle 1, -1\rangle$$

c) -**u**

$$-\mathbf{u} = -1\langle -5, 2\rangle = \langle 5, -2\rangle$$

d) 2**u** – **v**

$$2\mathbf{u} - \mathbf{v} = 2\langle -5, 2\rangle - \langle 6, -3\rangle$$
$$= \langle -10, 4\rangle - \langle 6, -3\rangle$$
$$= \langle -16, 7\rangle$$

Properties of Vector Addition and Scalar Multiplication

Let **u**, **v,** and **w** be vectors and let c and d be scalars. Then the following properties are true.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$
9. $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$

Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector **v**. To do this, divide the vector by its magnitude.

Example: If a vector is 4 units long, divide the vector by 4 so that it is now 1 unit long.

Definition: The vector **u** is the unit vector in the direction of **v** if

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

Example: Find a unit vector in the direction of **v** = $\langle -8, 6\rangle$.

$$\|\mathbf{v}\| = \sqrt{(-8)^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$$

$$\frac{1}{10}\mathbf{v} = \frac{1}{10}\langle -8, 6\rangle = \left\langle \frac{-8}{10}, \frac{6}{10} \right\rangle = \left\langle \frac{-4}{5}, \frac{3}{5} \right\rangle$$

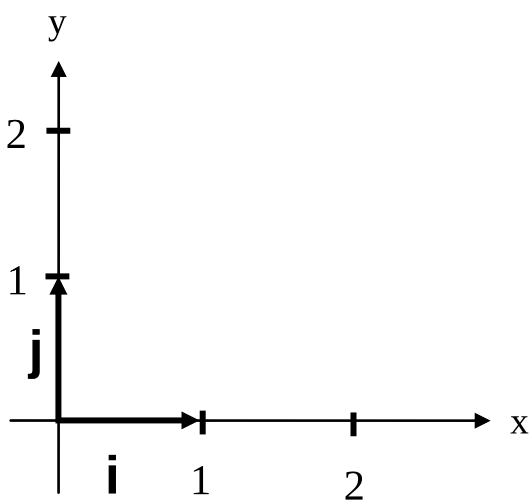
Example: Find the unit vector in the direction of **v** = $\langle 3, -5\rangle$

$$\|\mathbf{v}\| = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

$$\frac{1}{\sqrt{34}}\mathbf{v} = \frac{1}{\sqrt{34}}\langle 3, -5\rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{-5}{\sqrt{34}} \right\rangle = \left\langle \frac{3\sqrt{34}}{34}, \frac{-5\sqrt{34}}{34} \right\rangle$$

<u>Definition</u>: The unit vectors +1, 0 , and +0, 1 , are called the <u>standard unit vectors</u> and are denoted by

i = $\langle 1, 0 \rangle$ and **j** = $\langle 0, 1 \rangle$



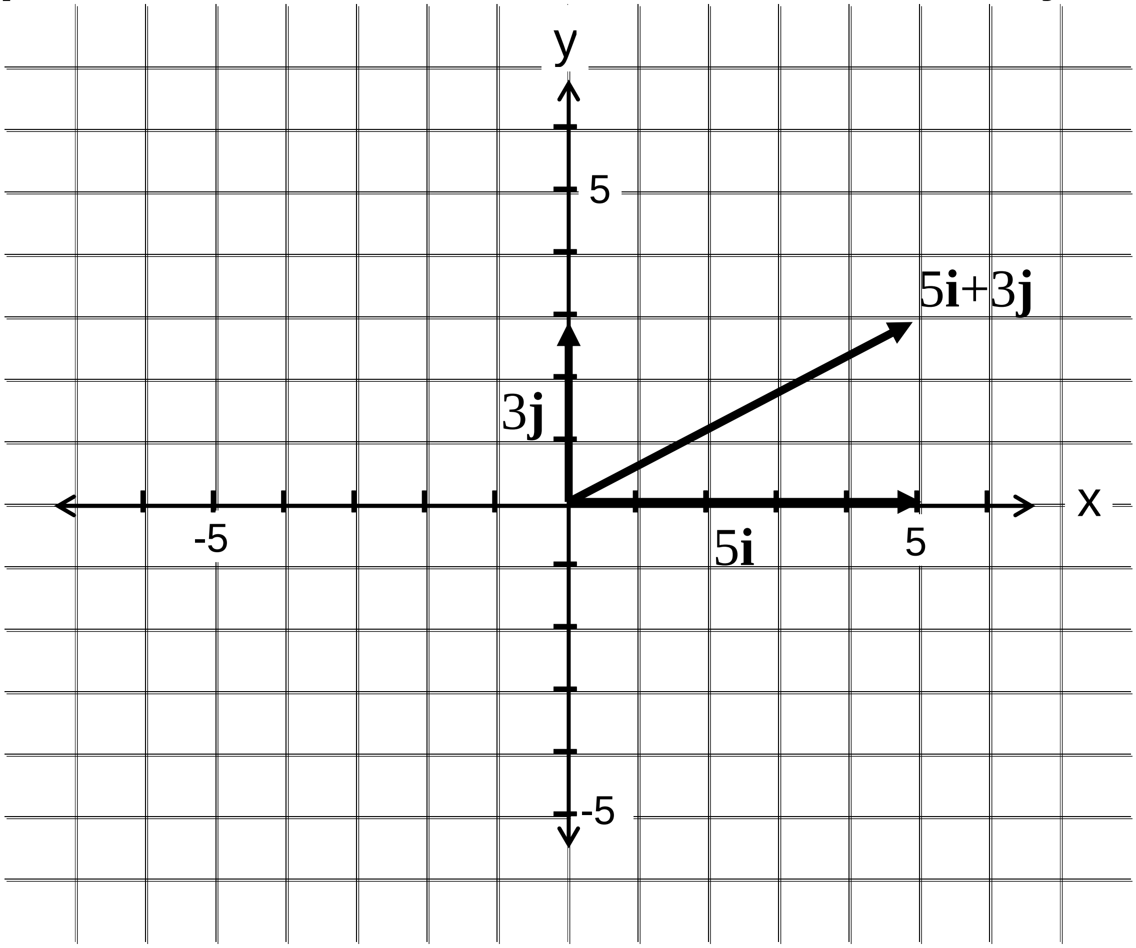
Standard unit vectors can be used to represent any vector **v** = $\langle v_1, v_2 \rangle$ in the following way:

v = $\langle v_1, v_2 \rangle$
= $v_1\langle 1,\ 0 \rangle$ + $\langle 0,\ 1 \rangle v_2$
= v_1**i** + v_2 **j**

The scalars v_1 and v_2 are called the <u>horizontal</u> and <u>vertical components of **v**</u>**.**

The vector sum v_1**i** + v_2 **j** is called a <u>linear combination of the vectors **i** and **j**</u>**.** Any vector in the plane can be expressed as a linear combination of **i** and **j.**

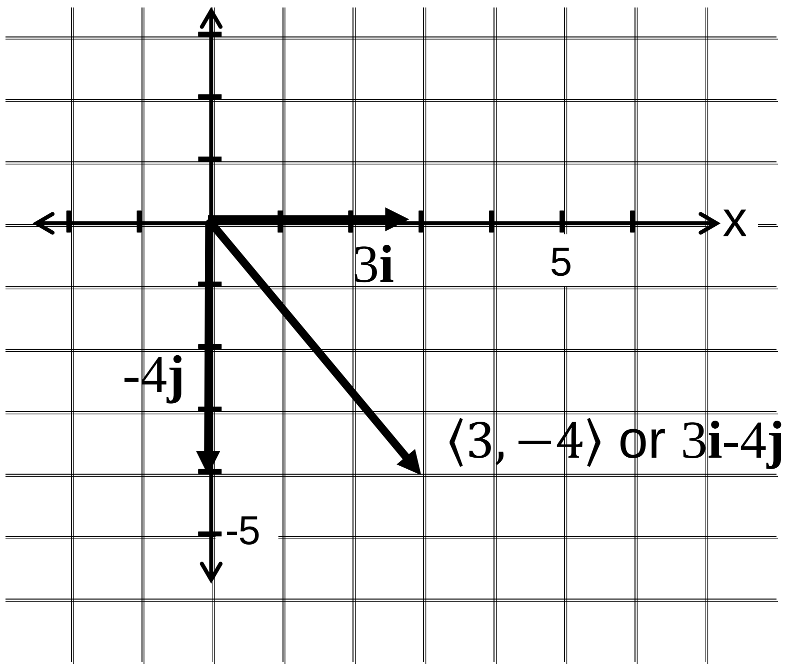
Example: The vector $\langle 5, 3 \rangle$ can be shown as 5**i**+3**j**.



Remember that the sum of 2 vectors is the vector that forms the diagonal of the parallelogram with the 2 vectors as the sides.

Example: Express **v** = $\langle 3, -4 \rangle$ as a linear combination of the vectors **i** and **j**.

v = $\langle 3, -4 \rangle$ = 3**i** – 4**j**



Example: Let **u** be a vector with initial point (2, -5) and terminal point (-1, 3). Write **u** as a linear combination of the standard unit vectors **i** and **j**.

First, write the component form of **u** by subtracting. (always begin with the terminal point)

$$\mathbf{u} = \langle -1 - 2, 3 - (-5) \rangle = \langle -3, 8 \rangle$$

Now write the component form as a linear combination of **i** and **j.**

$$\mathbf{u} = \langle -3, 8 \rangle = -3\mathbf{i} + 8\mathbf{j}$$

Example: Let **v** = 3**i** - 4**j** and **w** = 2**i** + 9**j**. Find **v** + **w**.

v + **w** = (3**i** - 4**j**) + (2**i** + 9**j**) = 5**i** + 5**j**

This is the vector $\langle 5, 5 \rangle$

Example: Let **v** = 3**i** - 4**j** and **w** = 2**i** + 9**j**. Find **v** - **w**.

v - **w** = (3**i** - 4**j**) - (2**i** + 9**j**) = 1**i** - 13**j**

This is the vector $\langle 1, -13 \rangle$

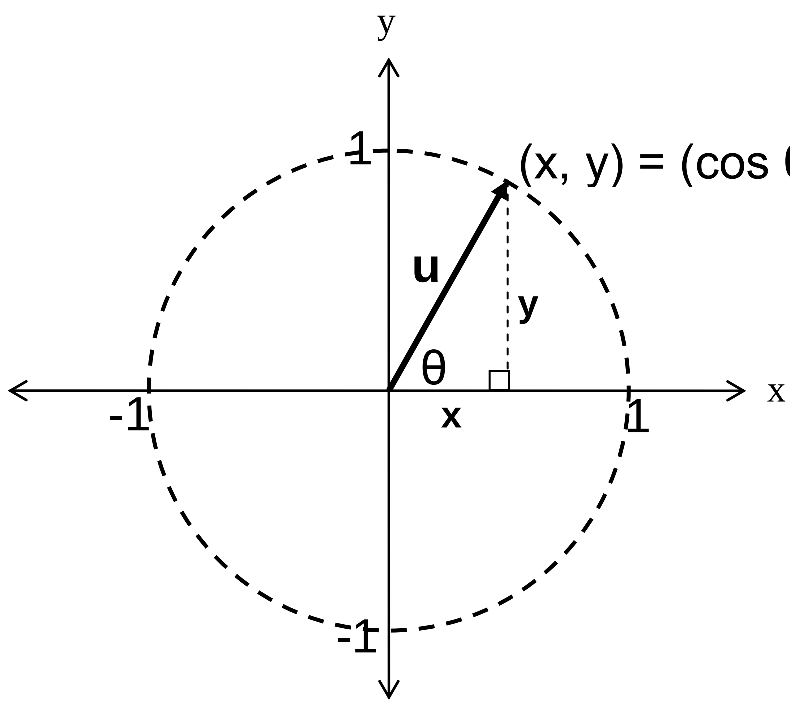
Example: Let **v** = 2**i** - 5**j** and **w** = -3**i** + 4**j**. Find 2**v** - 3**w**.

v - **w** = 2(2**i** -5**j**) -3(-3**i**+4**j**)
=4**i** – 10**j** +9**i** – 12**j**
=13**i**-22**j**

This is the vector $\langle 13, -22 \rangle$

Direction Angles

Consider the unit vector **u** as it is pictured below, with θ measured clockwise from the x-axis.



The unit vector **u** can be written as $\langle x, y\rangle$. Also,

$$\mathbf{u} = \langle x, y\rangle = \langle \cos\theta, \sin\theta\rangle = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$

The angle θ is called the direction angle of vector **u.**

If **v** is any vector that has angle θ as its direction vector, then **v** is a scalar multiple of **u.** Then

v = k**u**, where k is the magnitude (length) of **v**.

Example: Let **u** be the unit vector with directional angle θ.
Let **v** = 3i + 4j with directional angle θ.

We write **v** as a scalar multiple of **u**.

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$$

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

So, **v** = 5**u**

We can also write **v** = (5cos θ)**i** + (5sin θ)**j**

Since **v** = 3**i** + 4**j** *and*
v = (5cos θ)**i** + (5sin θ)**j**

then 3 = 5cos θ and 4 = 5sin θ.

↓ ↓

$\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$

Remember that 5 is the magnitude of v. That is, ||**v**|| = 5.

So we can write

$$\cos\theta = \frac{3}{\|v\|} \quad \text{and} \quad \sin\theta = \frac{4}{\|v\|}$$

Find tan θ.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

Notice how the tangent relates to our vector **v** = 3**i** + 4**j**.
This will happen with ALL vectors of the form a**i** + b**j**.

Definition: For any vector **v** = a**i** + b**j** with direction angle θ, the following are true:

$$\cos\theta = \frac{a}{\|v\|} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sin\theta = \frac{b}{\|v\|} = \frac{b}{\sqrt{a^2+b^2}}$$

$$\tan\theta = \frac{b}{a}$$

Example: Find the directional angle for **u** = 3**i** + 3**j**.

$$\tan\theta = \frac{b}{a} = \frac{3}{3} = 1$$

Since tangent = 1 at 45°, and u is in quadrant 1, we must have θ = 45°.

Example: Find the direction angle for **v** = 2**i** – **j**.

$$\tan\theta = \frac{b}{a} = \frac{-1}{2}$$

Since this is not a known angle, take the arctan of both sides. (Remember that the angle we get will be between -90° and 90° because that is the range of arctan.)

$$\arctan(\tan\theta) = \arctan(\frac{-1}{2})$$

$$\theta \approx -27°$$

Since (2, -1) is in quadrant 4, we use the measure of -27° but we must write it as a positive angle, since direction angles are measured clockwise (which makes them positive).

$\theta = 360° - 27° = 333°$.

Example: Let **v** = - 4**i** + 5**j**. Find the direction angle for **v**.

$$\tan\theta = \frac{b}{a} = \frac{5}{-4}$$

$$\arctan(\tan\theta) = \arctan(\frac{5}{-4})$$

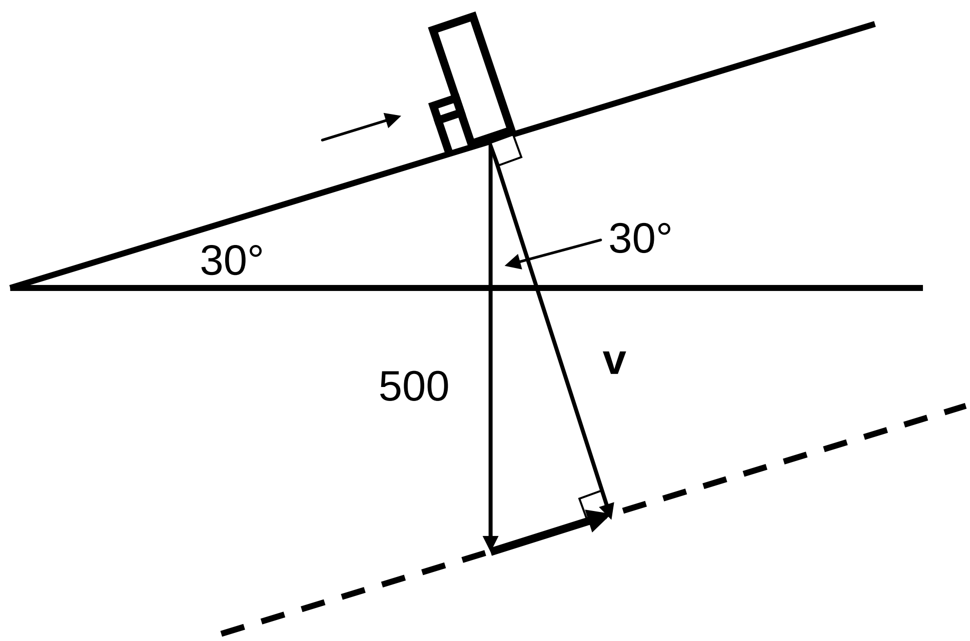
$$\theta \approx -51°$$

Since **v** is in Quadrant 2, we use the reference angle of 51° to find θ.

θ = 180°- 51° = 129°

Applications of Vectors (optional)

Example: A piano weighing 500 lb is being pushed up a ramp into the back of a truck. The ramp is a board that can support 450 lb and makes a 30° angle with the horizontal. Will the ramp support the piano?



Note: We know that both angles are 30° because of parallel lines and complementary angles.

We need to find the magnitude of the force vector **v**, which is perpendicular to the board.

Using trigonometry, we know that

$$\cos 30° = \frac{\|\mathbf{v}\|}{500}$$

Multiply both sides by 500 to get

$$\|\mathbf{v}\| = 500\cos 30° \approx 433 \text{ pounds}$$

Since this is less than 450 pounds, the board will support the piano.