Vectors and Dot Products

When we do vector addition or scalar multiplication, the answer is a vector. When we do the dot product, the answer will be a scalar (real number), not a vector.

<u>**Definition**</u>: The dot product of **u** $= \langle u_1, u_2 \rangle$ and **v** $= \langle v_1, v \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Example: Find the dot product of the following:

a) $\langle 5, -4 \rangle \cdot \langle 9, -2 \rangle$

answer: (5)(9) + (-4)(-2) = 45 + 8 = 53

b) $\langle 4, 3 \rangle \cdot \langle -2, -5 \rangle$

answer: (4)(-2) + (3)(-5) = -8 + -15 = -23

c) $\langle 0, -6 \rangle \cdot \langle 3, -7 \rangle$

answer: (0)(3) + (-6)(-7) = 0 + 42 = 42

Example: If **v** = ⟨3, 4⟩ find **v** · **v**.

$\langle 3, 4\rangle \cdot \langle 3, 4\rangle = 3^2 + 4^2 = 9 + 16 = 25$

This is $\|\mathbf{v}\|^2$.

$$\|v\| = \sqrt{3^2 + 4^2}$$
$$\|v\|^2 = \left(\sqrt{3^2 + 4^2}\right)^2$$
$$\|v\|^2 = 3^2 + 4^2$$

Properties of Dot Products

Let **u**, **v**, and **w** be vectors in the plane or in space and let *c* be a scalar.

1. **u** · **v** = **v** · **u**
2. **0** · **v** = 0
3. **u** · (**v** + **w**) = **u** · **v** + **u** · **w**
4. **v** · **v** = $\|\mathbf{v}\|^2$
5. *c*(**u** · **v**) = *c***u** · **v** = **u** · *c***v**

Example: Let **u** = $\langle 2, 6\rangle$ = $\langle -1, 5\rangle$, and **w** = $\langle -3, 1\rangle$. Find the following.

a) u · (**v** + **w**)

$\langle 2, 6\rangle \cdot (\langle -1, 5\rangle + \langle -3, 1\rangle) = \langle 2, 6\rangle \cdot \langle -4, 6\rangle = -8 + 36 = 28$

or

u · (**v** + **w**) = **u**· **v** + **u**· **w**

$= \langle 2, 6\rangle \cdot \langle -1, 5\rangle + \langle 2, 6\rangle \cdot \langle -3, 1\rangle.$

=(-2+30) + (-6+6)

=28 + 0

=28

b) (**u** · **v**)**w**

u · **v** = 2(-1) + 6(5) = 28

28**w** = $28\langle -3, 1\rangle. = \langle -84, 28\rangle$

c) **u** · 2**v**

$\langle 2, 6\rangle \cdot 2\langle -1, 5\rangle = \langle 2, 6\rangle \cdot \langle -2, 10\rangle$

= 2(-2) + 6(10)

= -4 + 60

= 56

Example: The dot product of **u** with itself is 6. What is the magnitude of **u**?

$\mathbf{u} \cdot \mathbf{u} = 6$, but $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$ so $6 = ||\mathbf{u}||^2$

$$6 = \|\mathbf{u}\|^2$$
$$\sqrt{6} = \sqrt{\|\mathbf{u}\|^2}$$
$$\|\mathbf{u}\| = \sqrt{6}$$

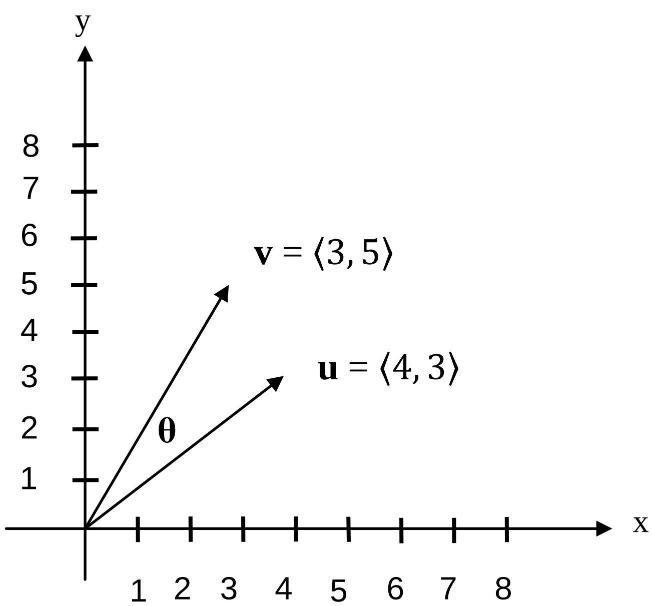
The Angle Between Two Vectors

Definition: The angle between two nonzero vectors is the angle θ between their respective standard position vectors, where $0 \le \theta \le \pi$.

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

Example: Find the angle between **u** = $\langle 4, 3\rangle$ and **v** = $\langle 3, 5\rangle$



$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

$$\cos\theta = \frac{\langle 4,3\rangle\cdot\langle 3,5\rangle}{\|\langle 4,3\rangle\|\,\|\langle 3,5\rangle\|}$$

$$\cos\theta = \frac{27}{5\sqrt{34}}$$

Using the arccosine function on a graphing calculator ($\cos^{-1}$), you find that θ ≈ 22.2°.

Example: Find the angle between **u** = $\langle -2, 3\rangle$ and **v** = $\langle 1, -5\rangle$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

$$\cos\theta = \frac{\langle -2,3\rangle\cdot\langle 1,-5\rangle}{\|\langle -2,3\rangle\|\,\|\langle 1,-5\rangle\|}$$

$$\cos\theta = \frac{-17}{\sqrt{13}\sqrt{26}}$$

$$\cos^{-1}\left(\frac{-17}{13\sqrt{2}}\right) = 157.6°$$

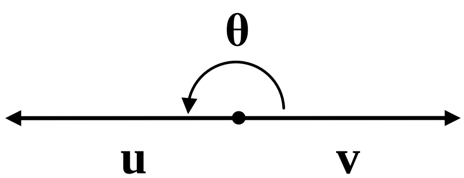
<u>Alternative form of Dot Product</u>

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

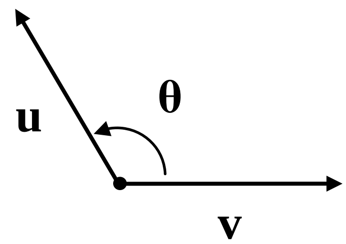
Since ||**u**||and ||v|| are always positive, then the dot product and cos θ must have the same sign.

positive dot product → positive cosine → acute angle
negative dot product → negative cosine → obtuse angle

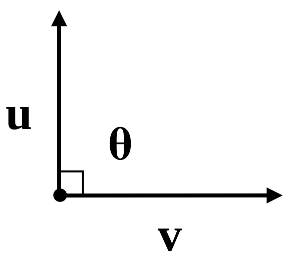
5 Possibile Orientation of vectors:



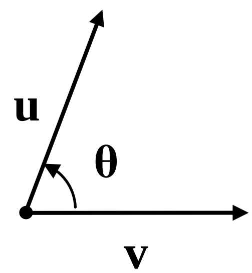
$\theta = \pi$
$\cos \theta = -1$
Opposite Direction



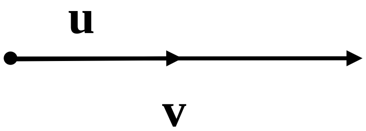
$\frac{\pi}{2} < \theta < \pi$
$-1 < \cos \theta < 0$
Obtuse Angle



$\theta = \frac{\pi}{2}$
$\cos \theta = 0$
90° Angle



$0 < \theta < \frac{\pi}{2}$
$0 < \cos \theta < 1$
Acute Angle



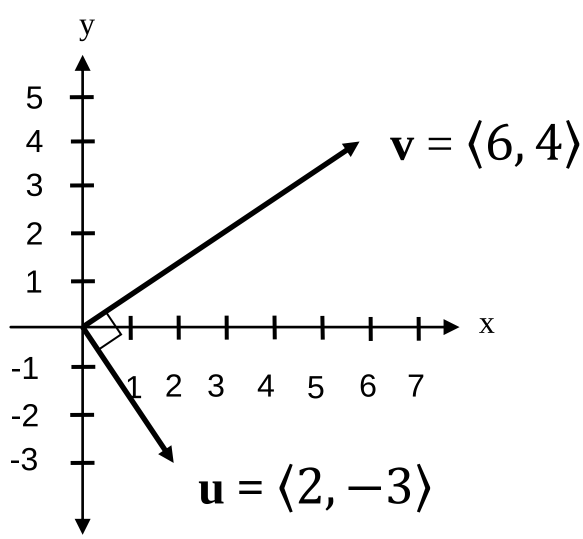
$\theta = 0$
$\cos \theta = 1$
Same Direction

Notice that for the 90° angle has cosine = 0. That means

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$
$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|(0)$$
$$\mathbf{u}\cdot\mathbf{v} = 0$$

Definition: The vectors **u** and **v** are <u>orthogonal</u> if u · v = 0.

Look at vectors **u** = $\langle 2, -3\rangle$ and **v** = $\langle 6, 4\rangle$



u · **v** = 6(2) + 4(-3) = 12 + (-12) = 0

Example: Are the vectors **u** = $\langle -4, 2\rangle$ and **v** = $\langle 1, 2\rangle$ orthogonal?

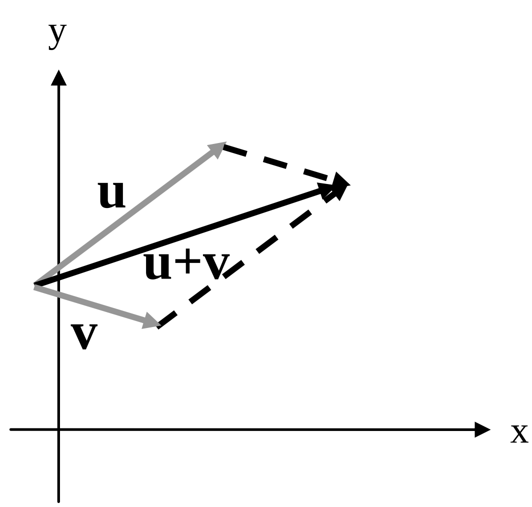
u · **v** = -4(1) + 2(2) = 0. So, **u** and **v** are orthogonal.

Example: Are the vectors **u** = $\langle 3, 4\rangle$ and **v** = $\langle 1, -5\rangle$ orthogonal?

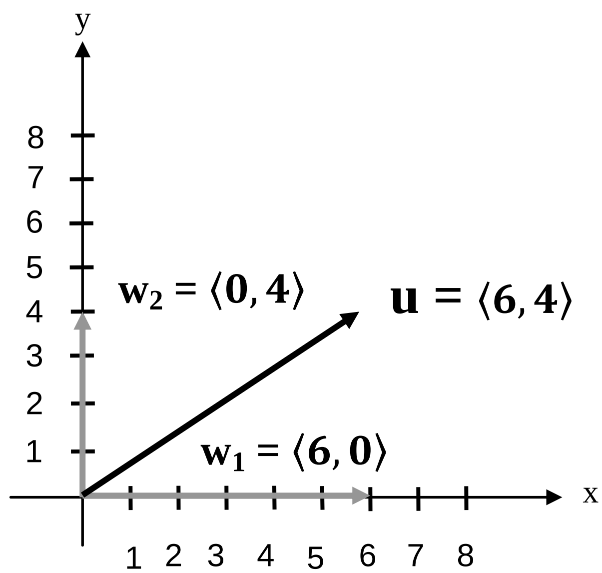
u · **v** = 3(1) + 4(-5) = -17. So, **u** and **v** are not orthogonal.

Finding Vector Components

We already know how to add 2 vectors to get a resultant vector. The resultant vector (the sum) is the diagonal of the parallelogram formed by the vectors.



What if the vectors are orthogonal?



We see that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$.

This is called <u>decomposing a vector into the sum of two vector components</u>. Many times in physics and engineering, a vector needs to be decomposed into vector components.