Matrices and Systems of Equations

Matrices

A matrix is a rectangular array of real numbers.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- We will use the double subscript notation for each element of the matrix. For example, the element of matrix A in the ith row and the jth column is denoted by a_{ij}.
- The dimensions of a matrix are given as *rows* x *columns.* This is read *"m* by *n."*
- A matrix is square if it is n x n; we say it has order n.
- The main diagonal of a square matrix is all the elements a_{ij} for which $i = j$.

Example: Determine the dimensions of each matrix.

a) $\begin{bmatrix} 3 \end{bmatrix}$

answer: 1 x 1

b) $\begin{bmatrix} 3 & -7 & \frac{2}{3} \end{bmatrix}$

answer: 1 x 3

c) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

answer: 2 x 2

d) $\begin{bmatrix} 1 & 5 & 3 \\ 9 & 5 & 0 \\ -5 & 3 & 9 \\ -8 & 4 & 8 \end{bmatrix}$

answer: 4 x 3

A matrix that has only one row is called a <u>row matrix</u>.

A matrix that has only one column is called a <u>column matrix</u>.

Look at the system
$$\begin{cases} x + y + 2z = 3 \\ 3x + 4y + 4z = 9 \\ 5x + 2y + 15z = 13 \end{cases}$$

If we write the system of equations without the variables, addition signs, and equal signs, and put it into a matrix, we get what is called an <u>augmented matrix</u>.

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 & 9 \\ 5 & 2 & 15 & 13 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 & 9 \\ 5 & 2 & 15 & 13 \end{array}\right] \text{ or } \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 & 9 \\ 5 & 2 & 15 & 13 \end{bmatrix}$$

(Different textbooks will write them in various ways.)

The <u>coefficient matrix</u> would be

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 4 \\ 5 & 2 & 15 \end{bmatrix}$$

<u>Note</u>: Any time a term is missing from a system of equations, a zero is put in its place in the matrix.

Elementary Row Operations

We need to translate the **elementary row operations** from our last unit into the language of matrices. Note that these operations will transform the augmented matrix of a system of equations into the augmented matrix of an equivalent system of equations.

If one matrix has been obtained from another matrix by using elementary row operations, then the two matrices are said to row-equivalent. Here are the operations.

Elementary Row Operations

1. Interchange two rows.
2. Replace any row by a nonzero multiple of itself.
3. Replace any row by the sum of itself and a multiple of any other row in the matrix.

We use the matrix version of the Gaussian Elimination method to solve systems of linear equations.

Example: Solve the system using the matrix version of Gaussian Elimination.

$$\begin{cases} x + y + 2z = 3 \\ 3x + 4y + 4z = 9 \\ 5x + 2y + 15z = 13 \end{cases}$$

Solution: Start with the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 & 9 \\ 5 & 2 & 15 & 13 \end{array}\right]$$

Multiply the 1st row by -3 and add to the 2nd row, replacing the 2nd row with your sum equation.

$$(-3)\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 & 9 \\ 5 & 2 & 15 & 13 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 5 & 2 & 15 & 13 \end{array}\right]$$

$$\begin{array}{ccc|c} -3 & -3 & -6 & -9 \end{array}$$

Multiply the 1st row by -5 and add to the 3rd row, replacing the 3rd row with your sum equation.

$$(-5)\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 5 & 2 & 15 & 13 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -3 & 5 & -2 \end{array}\right]$$

$$\begin{array}{ccc|c} -5 & -5 & -10 & -15 \end{array}$$

Multiply the 2nd row by 3 and add to the 3rd row, replacing the 3rd row with your sum equation.

$$(3)\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -3 & 5 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$$\begin{array}{ccc|c} 0 & 3 & -6 & 0 \end{array}$$

Replace the bottom row with -1 times itself.

$$\begin{array}{c} \\ \\ (-1) \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This final matrix is said to be in <u>row-echelon form</u>.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix is in row-echelon form if it has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. The first nonzero element of any row is 1, called a leading 1.
3. For two successive nonzero rows, the leading 1 in the upper row is farther to the left than the leading 1 in the lower row.

A matrix is in reduced row-echelon form if the column with a leading 1 has zeros everywhere else in that column.

Example: Continue working with the above augmented matrix until it is in reduced row-echelon form.

Multiply the 3^{rd} equation by 2 and add to the 2^{nd} equation, replacing the 2^{nd} equation with the sum.

$$\begin{array}{r} \\ \\ (2) \end{array}\left[\begin{array}{ccc:c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\hookrightarrow \begin{array}{ccc:c} 0 & 0 & 2 & 4 \end{array}$$

Multiply the 3rd equation by -2 and add to the 1st equation, replacing the 1st equation with the sum.

$$\begin{array}{r}
\\
\\
(-2)
\end{array}\left[\begin{array}{ccc:c}
1 & 1 & 2 & 3 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2
\end{array}\right] \to \left[\begin{array}{ccc:c}
1 & 1 & 0 & -1 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2
\end{array}\right]$$

$$\hookrightarrow \begin{array}{ccc:c} 0 & 0 & -2 & -4 \end{array}$$

Multiply the 2nd equation by -1 and add to the 1st equation, replacing the 1st equation with the sum.

$$\begin{array}{r}
\\
(-1)
\\
\end{array}\left[\begin{array}{ccc:c}
1 & 1 & 0 & -1 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2
\end{array}\right] \to \left[\begin{array}{ccc:c}
1 & 0 & 0 & -5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2
\end{array}\right]$$

$$\hookrightarrow \begin{array}{ccc:c} 0 & -1 & 0 & -4 \end{array}$$

The final matrix is in reduced row-echelon form.

Look at the row-echelon form and the reduced row-echelon form. Use back substitution to solve the systems that each represents.

First the row-echelon form:

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \begin{cases} x + y + 2z = 3 \\ \qquad y - 2z = 0 \\ \qquad\qquad z = 2 \end{cases}$$

$$z = 2$$

$$\begin{aligned} y - 2z &= 0 \\ y - 2(2) &= 0 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} x + y + 2z &= 3 \\ x + 4 + 2(2) &= 3 \\ x &= -5 \end{aligned}$$

The solution is (-5, 4, 2).

Now the reduced row-echelon form.

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \begin{cases} x = -5 \\ y = 4 \\ z = 2 \end{cases}$$

The solution is (-5, 4, 2).

**Both matrices gave us the same solution.

Solving a system of equations by <u>Gaussian elimination</u> is taking the augmented matrix of the system of equations and transforming it into <u>row-echelon form</u>, and then using back-substitution to find the values of the variables.

Solving a system of equations by <u>Gauss-Jordan elimination</u> is taking the augmented matrix of the system of equations and transforming it into <u>reduced row-echelon form</u>. The last column gives the values of the variables.

<u>Tip</u>: Work column by column. Get the first column the way you want it, then move to the second column, and so on.

Example: Which of the following matrices are in reduced row-echelon form? If it is not in reduced row-echelon form, state whether it is in row-echelon form.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$

reduced row-echelon form

reduced row-echelon form

c) $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

just row-echelon form *reduced row-echelon form*

Example: Solve the system by Gaussian elimination.

$$\begin{cases} x + 2y + 3z = 4 \\ 3x + 7y + 11z = 15 \\ -2x - 2y - z = 0 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 11 & 15 \\ -2 & -2 & -1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is (0, -1, 2).

Example: Solve by Gauss-Jordan elimination.

$$\begin{cases} 2x + 4y = 6 \\ 3x + 7y = 5 \end{cases}$$

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -4 \end{bmatrix}$$

The solution is (11, -4).

Example: Solve the system by Gauss-Jordon elimination.

$$\begin{cases} x + 2y + 3z = 1 \\ x + 3y + z = -1 \\ 3x + 7y + 7z = 6 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 1 & -1 \\ 3 & 7 & 7 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This system has no solution.

Example: Solve the system by Gauss-Jordon elimination.

$$\begin{cases} x + 2y - 3z = 2 \\ 2x + 5y - 6z = 1 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 2 & 5 & -6 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Note: With only 2 equations, it is impossible to get rid of both y and z in the top equation.

Let z = a. Then y = -3 and x – 3z = 8, which means x = 3a + 8 when we substitute a for z.

The solution to the system is (3a+8, -3, a).

Example: Solve the system.

$$\begin{cases} 2x - y + 3z = 1 \\ x + 2y - 4z = -6 \\ -2x + 3y - z = 13 \end{cases}$$

The solution is (-2, 4, 3).

Matrix Operations on the Graphing Calculator

Entering a Matrix

1. Press [2nd] [MATRX] [EDIT]
2. Choose [A] to enter your matrix as matrix A.
3. On the top of the edit screen you must enter the dimensions (i.e. order) for your matrix. Move the cursor to each spot and enter the number of rows and then columns.
4. Move the cursor down to the row 1, column 1 entry spot. You will see the row, column reference at the bottom of the screen. Type in your entry. Press [ENTER]. Your cursor goes to the next element's spot.
5. Continue entering all of the elements of the matrix. When finished, press [2nd] [Quit].
6. To see the matrix on your main screen, press [2nd][MATRX] [A] [ENTER].

Example: Enter the augmented matrix into your graphing calculator as matrix A for the following system.

$$\begin{cases} 2x+4y=6 \\ 3x+7y=5 \end{cases}$$

Changing a Matrix to Row-Echelon Form

1. Press [2nd] [MATRX] [MATH] [ref( ]. ("ref" stands for "row-echelon form.")
2. On your main screen you will see "ref(". You need to tell the calculator the name of the matrix that you want it to put in row-echelon form. To do this, press [2nd] [MATRX] [A] to enter matrix A. Press [ENTER].

Note: Row-echelon form matrices are not unique. It depends on the sequence of row operations chosen.

Example: Using the above system and matrix entered into your calculator as matrix A, find the row-echelon form on your graphing calculator.

$$\begin{bmatrix} 1 & 2.333... & 1.666... \\ 0 & 1 & -4 \end{bmatrix}$$

Notice how it writes the fractions as decimals. To solve this system, we would have to back-substitute.

Changing a Matrix to Reduced Row-Echelon Form

1. Press [2nd] [MATRX] [MATH] [rref( ]. ("rref" stands for "reduced row-echelon form.")
2. On your main screen you will see "rref(". You need to tell the calculator the name of the matrix that you want it to put in row-echelon form. To do this, press [2nd] [MATRX] [A] to enter matrix A. Press [ENTER].

Note: Reduced Row-echelon form matrices are unique. They will be the same regardless of the choices made in row operations.

Example: Using the above system and matrix entered into your calculator as matrix A, find the reduced row-echelon form on your graphing calculator.

$$\begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -4 \end{bmatrix}$$

This tells us that the solution to the system is (11, -4).