The Inverse of a Square Matrix

If a is real number, then its <u>multiplicative inverse</u> is a^{-1} and the following is true:

$$aa^{-1} = 1$$

The definition of the multiplicative inverse of a matrix is similar.

<u>Definition</u>: <u>The Inverse of a Square Matrix</u>**:**

Let A be an *n x n* matrix and let I_n be the *n x n* identity matrix. If there exists matrix A^{-1} such that

$$A A^{-1} = I_n = A^{-1} A$$

then A^{-1} is called the <u>inverse</u> of A. The symbol A^{-1} is read "A inverse."

Example: Show that B is the inverse of A, where

$$A = \begin{bmatrix} -5 & 6 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -6 \\ 1 & -5 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} -5 & 6 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -6 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -6 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -5 & 6 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding Inverse Matrices

Definition: If matrix A has an inverse, A is called invertible (or nonsingular). Otherwise, A is called singular.

Example: Find the inverse of $A = \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}$.

We need to find the matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AB = I_2$.

$$AB = \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4a+5c & 4b+5d \\ 7a+9c & 7b+9d \end{bmatrix}$$

If B is the inverse, then we must have

$$AB = \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4a+5c & 4b+5d \\ 7a+9c & 7b+9d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$4a+5c=1$$
$$7a+9c=0$$
$$4b+5d=0$$
$$7b+9d=1$$

We can put these together into 2 systems:

$$\begin{cases} 4a+5c=1 \\ 7a+9c=0 \end{cases} \Rightarrow \begin{bmatrix} 4 & 5 & 1 \\ 7 & 9 & 0 \end{bmatrix} \text{ and } \begin{cases} 4b+5d=0 \\ 7b+9d=1 \end{cases} \Rightarrow \begin{bmatrix} 4 & 5 & 0 \\ 7 & 9 & 1 \end{bmatrix}$$

Solving these gives us $a=9, b=-5, c=-7, d=4$.

Thus, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$

Therefore, the inverse B, of A is $\begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$

Notice that both of the systems had the same coefficient matrix A. Rather than solving each system separately, we can solve them simultaneously by adjoining the last column of each system to the coefficient matrix and applying Gauss-Jordon elimination.

This looks like: $\begin{bmatrix} 4 & 5 & 1 & 0 \\ 7 & 9 & 0 & 1 \end{bmatrix}$ or $\left[\begin{array}{cc:cc} 4 & 5 & 1 & 0 \\ 7 & 9 & 0 & 1 \end{array}\right]$

If you take this matrix, represented as [A: I], and apply Gauss-Jordon elimination, you will end up with the matrix [I : A^{-1}].

$$\left[\begin{array}{cc:cc} 4 & 5 & 1 & 0 \\ 7 & 9 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc:cc} -28 & -35 & -7 & 0 \\ 28 & 36 & 0 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cc:cc} 4 & 5 & 1 & 0 \\ 0 & 1 & -7 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc:cc} 4 & 5 & 1 & 0 \\ 0 & -5 & 35 & -20 \end{array}\right] \Rightarrow \left[\begin{array}{cc:cc} 4 & 0 & 36 & -20 \\ 0 & 1 & -7 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 9 & -5 \\ 0 & 1 & -7 & 4 \end{array}\right]$$

So, $\left[\begin{array}{cc:cc} 4 & 5 & 1 & 0 \\ 7 & 9 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 9 & -5 \\ 0 & 1 & -7 & 4 \end{array}\right]$

A I I A^{-1}

Finding an Inverse Matrix

Let A be a square matrix of order n.

1. Form the n x $2n$ matrix $[A : I_n]$.
2. Transform the matrix into reduced row-echelon form.
3. If this new matrix is of the form $[I_n : B]$, then A is invertible and $B = A^{-1}$.

Example: Find the inverse matrix for $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 2 \\ 2 & 3 & 6 \end{bmatrix}$.

Solution:

$$(-1)\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$-R_2 + R_1$

$$(-2)\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 \\ 0 & 3 & 2 & 0 & -2 & 1 \end{array}\right]$$

$-2 R_1 + R_3$

$$\begin{array}{r} \\ (-3) \\ (2) \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 \\ 0 & 3 & 2 & 0 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & -1 & 0 \\ 0 & 0 & -2 & -3 & -1 & 2 \end{array}\right]$$

$-3R_2 + 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 2 & 2 & 1 & -1 & 0 \\ 0 & 0 & -2 & -3 & -1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 2 & 0 & -2 & -2 & 2 \\ 0 & 0 & -2 & -3 & -1 & 2 \end{array}\right]$$

$R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & -2 & -3 & -1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 3/2 & 1/2 & -1 \end{array}\right]$$

½ R_2 - ½ R_3

$$A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & -1 & 1 \\ 3/2 & 1/2 & -1 \end{bmatrix}$$

Using a Graphing Calculator to Find Matrix Inverses

To find the inverse of the square matrix A, do the following:

1. Press [2nd] [MATRX] [A] [x-1] [ENTER].

    - If the matrix is invertible, it will show the inverse on the screen.
    - If the matrix is not invertible, it will show the error message “SINGULAR MAT” which means it is a singular matrix.

Example: Using a graphing calculator, find the inverse of

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$$

Solution: $$A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

Example: Using a graphing calculator, find the inverse of

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

Solution: The matrix does not have an inverse.

Example: Using a graphing calculator, find the inverse of

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} \quad \textit{solution}: \ B^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

The Inverse of a 2 x 2 Matrix

If A is a *2 x 2* matrix given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible only if ad – bc ≠ 0.

If ad – bc ≠ 0, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{\text{(downs) - (ups)}}$$

Exchange d and a, and change the signs on b and c.

Note: This formula works *only* for 2 x 2 matrices.

Example: Find the inverse of $B = \begin{bmatrix} 3 & 9 \\ -2 & -7 \end{bmatrix}$

$$B^{-1} = \frac{1}{(-21)-(-18)}\begin{bmatrix} -7 & -9 \\ 2 & 3 \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} -7 & -9 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7/3 & 3 \\ -2/3 & -1 \end{bmatrix}$$

Example: Find the inverse of $A = \begin{bmatrix} 3 & 2 \\ 9 & 5 \end{bmatrix}$

$$A^{-1} = \frac{1}{15-18}\begin{bmatrix} 5 & -2 \\ -9 & 3 \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} 5 & -2 \\ -9 & 3 \end{bmatrix} = \begin{bmatrix} -5/3 & 2/3 \\ 3 & -1 \end{bmatrix}$$

Example: Find the inverse of $C = \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix}$

Since (-2)(-6) – (4)(3) = 0, this matrix is singular.

Systems of Linear Equations

Remember that we can represent a system of equations by the matrix equation AX = B, where A is the coefficient matrix and X and B are column matrices.

A · X = B

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1x + 2y + 3z \\ 5x + 6y + 7z \\ 9x + 10y + 11z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

$$\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 9x + 10y + 11z = 12 \end{cases}$$

Using what we know about the inverses of matrices, we can solve the matrix equation similarly as we would an algebraic equation.

Algebra	Matrices
$ax = b$	$AX = B$
$a^{-1}ax = a^{-1}b$	$A^{-1}AX = A^{-1}B$
$1x = a^{-1}b$	$IX = A^{-1}B$
$x = a^{-1}b$	$X = A^{-1}B$

Note: Because of the nature of matrix multiplication, you must multiply the inverse on the left of each side.

If the coefficient matrix of a square system is invertible, then the system has a unique solution (instead of infinitely many solutions or no solution.)

A System of Equation with a Unique Solution

If A is an invertible matrix, the system of linear equations represented by $AX = B$ has a unique solution given by

$$X = A^{-1}B$$

Example: Solve the following system of equations.

$$\begin{cases} 2x - 5y = 8 \\ 3x - 8y = 1 \end{cases}$$

<u>*Solution*</u>:

Write as the matrix equation $\begin{bmatrix} 2 & -5 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

Rewrite as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & -8 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 1 \end{bmatrix}$

Find $\begin{bmatrix} 2 & -5 \\ 3 & -8 \end{bmatrix}^{-1}$

We get $\begin{bmatrix} 2 & -5 \\ 3 & -8 \end{bmatrix}^{-1} = \dfrac{1}{-1}\begin{bmatrix} -8 & 5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ 3 & -2 \end{bmatrix}$

Solve $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & -8 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 59 \\ 22 \end{bmatrix}$

Therefore the solution is x = 59 and y = 22.

Example: Solve the following system of equations.

$$\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

where $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

So $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

The solution is x = 6 and y = 3.

Solving a System of Equations using a Matrix Equation using a Graphing Calculator

When solving the matrix equation AX = B by using the matrix equation $X = A^{-1}B$, do the following:

2. Enter matrices A and B in the matrix editor using [2nd] [MATRX] [EDIT].
3. Press [2nd] [MATRX] [A] [x^{-1}] [2nd] [MATRX] [B] [ENTER].

The screen will display the solution matrix X.

- If the matrix A is not invertible, or the matrix dimensions are incompatible for matrix multiplication, you will get an error message.

Example: Solve the following system using matrix operations on a graphing calculator.

$$\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$$

Solution: $$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

Enter $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$ in the matrix editor.

Enter [A] [x^{-1}] [B] and get $X = \begin{bmatrix} 3 \\ 8 \\ -11 \end{bmatrix}$

Therefore, the solution is x = 3, y = 8, and z = -11.