The Determinant of a Square Matrix

Every square matrix can be associated with a real number called its determinant.

The Determinant of a 2 x 2 Matrix

Definition of the Determinant of a 2 x 2 Matrix

The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(The vertical bars, when used in the context of matrices, represents determinants and not absolute value.)

*Note: This is the (downs) - (ups) number that we used when using the formula for finding inverses of 2 x 2 matrices.

Example: Find the determinant of the following matrices.

a) $A = \begin{bmatrix} -6 & 2 \\ 7 & 1 \end{bmatrix}$

solution: $\det(A) = -6 - 14 = -20$

b) $B = \begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix}$

solution: $\det(B) = 8 - 3 = 5$

c) $C = \begin{bmatrix} -2 & 8 \\ 3 & -12 \end{bmatrix}$

solution: $\det(C) = 24 - 24 = 0$

<u>The Determinant of a Square Matrix</u>

*The determinant of a matrix of order *1 x 1* is defined as the entry of the matrix. For instance, if A = [-3], then det(A)=-3.

*The determinant of a *2 x 2* matrix can be found by the method above.

*The determinant of a *3 x 3* matrix can be found by taking the sum of the products of the down diagonals minus the sum of the products of the up diagonals.

- The following example shows how to most easily find the sum of the diagonals.

Example: Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{bmatrix}$$

Solution: Write the matrix and then rewrite the first 2 columns after the matrix as follows:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ 3 & -1 \\ 4 & 1 \end{matrix}$$

Circle all of the down diagonals and find the products.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ 3 & -1 \\ 4 & 1 \end{matrix}$$

-1 16 3

The sum of the products of the down diagonals is 18.

Circle all of the up diagonals and find the products.

-4 2 6

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ 3 & -1 \\ 4 & 1 \end{matrix}$$

The sum of the products of the up diagonals is 4.

Then det(A) = 18 – 4 = 14.

*<u>Note</u>: As with the 2 x 2 determinants, you can think of this as the (downs) – (ups).

Example: Find the determinant of $B = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 3 & 4 \\ -1 & 1 & 5 \end{bmatrix}$

6 4 0

$$B = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 3 & 4 \\ -1 & 1 & 5 \end{bmatrix} \begin{matrix} 1 & 3 \\ 0 & 3 \\ -1 & 1 \end{matrix}$$

15 -12 0

det(B) = 3 -10= -7

Example: Find the determinant of $C = \begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 4 \\ 8 & 4 & 5 \end{bmatrix}$

-72 16 0

$$C = \begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 4 \\ 8 & 4 & 5 \end{bmatrix} \begin{matrix} 1 & 0 \\ -7 & 3 \\ 8 & 4 \end{matrix}$$

15 0 84

det(C) = 99+56= 155

Finding the Determinant of a Matrix Using a Graphing Calculator

1. Enter the matrix A in the matrix editor using [2nd] [MATRX] [EDIT].
2. Press [2nd] [MATRX] [MATH] [det].
3. Press [2nd] [MATRX] [A] [ENTER].

Example: Find the determinant of matrices A and B in the above examples.

det(A) = 14 and det(B) = -7

Minors and Cofactors

Definition: If A is a square matrix, the minor M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.

In other words, the minor of an element is the value of the determinant found by deleting the row and column of that element.

Example: Find 2 minors of the matrix $C = \begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix}$

The minor for 1 is $\begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$ =15 – 8 = 7

The minor for 0 is $\begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix} = \begin{vmatrix} -7 & 2 \\ 8 & 5 \end{vmatrix}$ = -35 – 16 = -51

Example: Find the minor of -3 in matrix C.

The minor for -3 is $\begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix} = \begin{vmatrix} -7 & 3 \\ 8 & 4 \end{vmatrix}$ = -28 – 24 = -52

Example: Find the minor of 3 in matrix C.

The minor for 3 is $\begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix} = \begin{vmatrix} 1 & -3 \\ 8 & 5 \end{vmatrix}$ = 5 + 24 = 29

<u>Definition</u>: If A is a square matrix, with <u>minor</u> M_{ij} of the entry a_{ij}, then the <u>cofactor</u> C_{ij} of the entry a_{ij} is

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

In other words, the cofactor of an element will either equal the minor of the element or be the opposite of the minor of the element.

Example: Using the minors from above, find the cofactors.

a) If M_{11} = 7, then cofactor $C_{11} = (-1)^{1+1}(7) = (-1)^2(7) = 7$

b) If $M_{12} = -51$, then

$$\text{cofactor } C_{12} = (-1)^{1+2}(-51) = (-1)^{3}(-51) = 51$$

c) If $M_{13} = -52$, then

$$\text{cofactor } C_{13} = (-1)^{1+3}(-52) = (-1)^{4}(-52) = -52$$

d) If $M_{22} = 29$, then

$$\text{cofactor } C_{22} = (-1)^{2+2}(29) = (-1)^{4}(29) = 29$$

***In general, if the sum of the row and column numbers is odd, then the cofactor is the opposite of the minor. Otherwise it is the same as the minor.

Sign Pattern for Cofactors

The + and – signs represent $(-1)^{i+j}$ for each element.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

Notice the checkerboard pattern.

Example: Use the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$ to find

a) The minor M_{13}.

$$M_{13} = \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$$

b) The cofactor C_{21}.

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} = (-1)(0-6) = (-1)(-6) = 6$$

c) The cofactor C_{33}.

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (1)(1-0) = (1)(1) = 1$$

Using Cofactors to find the Determinant of a Square Matrix

If *A* is a square matrix, then the determinant of *A* is the sum of the products of each element in any row (or column) of *A* and their cofactors.

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + ... + a_{1n}C_{1n}$$

This method of finding the determinant is called "expanding by cofactors."

Example: Find the determinant by expanding along the first row of matrix C.

$$C = \begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix}$$

We know the minors and cofactors of the first row from the example above:

- The minor of 1 is 7 and the cofactor is 7.
- The minor of 0 is -51 and the cofactor is 51.
- The minor of -3 is -52 and the cofactor is -52.

Then det(C)=1(7) + 0(51) + -3(-52) = 163

Example: Find the determinant by expanding along the first column of matrix C.

$$C = \begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix}$$

$$\det(C) = 1\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} - (-7)\begin{vmatrix} 0 & -3 \\ 4 & 5 \end{vmatrix} + 8\begin{vmatrix} 0 & -3 \\ 3 & 2 \end{vmatrix} =$$

1(7) + 7(12) + 8(9) = 163

Example: Find the determinant by expanding along the second row of matrix C.

$$C = \begin{bmatrix} 1 & 0 & -3 \\ -7 & 3 & 2 \\ 8 & 4 & 5 \end{bmatrix}$$

$$\det(C) = -(-7)\begin{vmatrix} 0 & -3 \\ 4 & 5 \end{vmatrix} + 3\begin{vmatrix} 1 & -3 \\ 8 & 5 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 8 & 4 \end{vmatrix} =$$

=7(12) + 3(29) – 2(4) = 163

When expanding along a row or column to find the determinant, think about the checkerboard to find out whether to add the element or subtract it. Remember that the signs alternate, so you really only need to find out the very first sign.

Example: Find the determinant of $A = \begin{bmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix} - (-)1\begin{vmatrix} 8 & 2 \\ 1 & 3 \end{vmatrix} + 5\begin{vmatrix} 8 & -3 \\ 1 & -2 \end{vmatrix}$$

= 2(-5) + 1(22) + 5(-13) = -53

Example: Find the determinant of $B = \begin{bmatrix} -1 & 3 & 4 \\ 3 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

<u>Hint</u>: If you pick a row or column that has a zero in it, there will be less calculations to make.

Expand along the 2nd row.

$$|B| = \begin{vmatrix} -1 & 3 & 4 \\ 3 & -2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = -3\begin{vmatrix} 3 & 4 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix}$$

= -3(7) – 2(-5) + 0 = -11

Example: Find the determinant of

$$D = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 2 & 1 & -2 & 1 \\ 0 & 2 & -1 & 0 \\ -3 & -2 & 2 & 1 \end{bmatrix}$$

When solving by expanding with cofactors, choose row 3 because it contains the most zeros.

$$\det(D) = 0\begin{vmatrix} 0 & 3 & 4 \\ 1 & -2 & 1 \\ -2 & 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 & 4 \\ 2 & -2 & 1 \\ -3 & 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ -3 & -2 & 2 \end{vmatrix}$$

2 of these elements are 0, so we end up with:

$$\det(D) = -2\begin{vmatrix} 1 & 3 & 4 \\ 2 & -2 & 1 \\ -3 & 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix}$$

Find each 3 x 3 determinant.

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & -2 & 1 \\ -3 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -2 \\ -3 & 2 \end{vmatrix}$$

= 1(-4) – 3(5) + 4(-2) = -27

$$\begin{vmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -3 & -2 \end{vmatrix}$$

= 1(3) – 0 + 4(-1) = -1

$$\det(D) = -2\begin{vmatrix} 1 & 3 & 4 \\ 2 & -2 & 1 \\ -3 & 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix} = -2(-27) - 1(-1) = 55$$

(You could have used “downs – ups” for these 2 determinants since they are 3 x 3).