Geometric Sequences and Series

A sequence whose consecutive terms have a common difference is called an arithmetic sequence. You subtract back to find the common difference.

A sequence whose consecutive terms have a common ratio is called a geometric sequence. You divide back to find the common ratio.

Definition of Geometric Sequence

A sequence is geometric if the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots a_n, \ldots$$

is geometric if there is a number r such that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = r, \quad r \neq 0$$

and so on. The number r is the common ratio of the geometric sequence.

Example: Find the common ratios in the following geometric sequences.

a) 3, 6, 12, 24, …

$$r = \frac{a_2}{a_1} = \frac{6}{3} = 2$$

b) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$$r = \frac{a_2}{a_1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

Example: Determine which of the following are geometric sequences and find the common ratio if they are.

a) 3, 6, 9, 12, 15,…

no

b) 2, 4, 8, 16, 32, …

yes, $r = 2$

c) 1, -1, 1, -1, 1, …

yes, $r = -1$

d) 4, 2, 1, ½, ¼ , …

yes, $r = \frac{1}{2}$

e) 2, 4, 16, 64, 256, …

no

<u>Examples of Geometric Sequences</u>

The sequence with *n*th term $a_n = 2^n$.

$2, 4, 8, 16 \ldots\ldots 2^n, \ldots \qquad r=2$

The sequence with *n*th term $a_n = 4(3^n)$.

$12, 36, 108, 324, \ldots.\ 4(3^n).,\ldots \qquad r = 3$

The sequence with *n*th term $a_n = \left(-\frac{1}{3}\right)^n$

$$-\frac{1}{3},\frac{1}{9},-\frac{1}{27},\frac{1}{81},\ldots,\left(-\frac{1}{3}\right)^n,\ldots \qquad r=-\frac{1}{3}$$

**Notice that the number being taken to the *n*th power is the common ratio.

The *n*th Term of a Geometric Sequence

The *n*th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where *r* is the common ratio between consecutive terms of the sequence.

Look at the pattern of a geometric sequence:

$$a_1 = a_1$$
$$a_2 = a_1 r$$
$$a_3 = a_2 r = (a_1 r)r = a_1 r^2$$
$$a_4 = a_3 r = (a_1 r^2)r = a_1 r^3$$
$$a_5 = a_4 r = (a_1 r^3)r = a_1 r^4$$
$$\vdots$$
$$a_n = a_1 r^{n-1}$$

1 less than the value of the index.

The *n*th term of an arithmetic sequence has the form

$$a_n = a_1 r^{n-1}$$

where *r* is the common ratio between consecutive terms of the sequence.

Example: Find the *n*th term of the geometric sequence with common ratio 2 and first term 5.

$$a_n = a_1 r^{n-1}$$

$$a_n = 5(2^{n-1})$$

Example: Find the 8th term of the geometric sequence if the first 2 terms are 15 and 12.

$$a_n = a_1 r^{n-1}$$

$$r = \frac{12}{15} = \frac{4}{5} \qquad \text{so} \qquad a_8 = 15\left(\frac{4}{5}\right)^7$$

$$a_8 = 3.145728$$

Example: Find the 20th term of the geometric sequence 1, 3, 9, 27,…

$$a_n = a_1 r^{n-1}$$

$$a_{20} = 1(3^{20-1})$$

$$a_{20} = 3^{19}$$

$$a_{20} = 1{,}162{,}261{,}467$$

Writing the Terms of a Geometric Sequence

You can find the terms of a geometric sequence given any 2 terms of the sequence.

Example: The 6th term of a geometric sequence is 384, and the 10th term is 6144. Write the first five terms of this sequence.

We have $a_6 = 384$ and $a_{10} = 6144$.

Remember that to get from one term to the next, we always multiply by *r*, so to get from the 6th term to the 10th term, we just multiply by 10 – 6 = 4 *r*'s (i.e. r^4). In other words,

$$a_{10} = a_6 r^4$$

If we substitute the values for a_{10} and a_6 we get

$$a_{10} = a_6 r^4$$
$$6144 = 384 r^4$$
$$r^4 = \frac{6144}{384} = 16$$
$$r = \pm 2$$

Dividing back from our 6th term gives us 12, 24, 48, 96,192 **<u>or</u>** -12, 24, -48, 96, -192 as the first 5 terms. (2 possible sequences)

Example: Find the positive *n*th term of the geometric sequence with 3rd term $\frac{16}{3}$ and 5th term $\frac{64}{27}$.

Solution: To get from the 3rd term to the 5th term, we need to start with the 3rd term and multiply it by 5 - 3 = 2 *r*'s (i.e. r^2).

$$a_5 = a_3 r^2$$

$$\frac{64}{27} = \frac{16}{3} r^2$$

$$r^2 = \frac{64}{27} \cdot \frac{3}{16} = \frac{4}{9}$$

$$r = \frac{2}{3}$$

To find the *n*th term of the sequence, take the general formula and put in the values for one of the given terms in order to solve for a_1.

$$a_n = a_1 r^{n-1}$$

$$a_5 = a_1 \left(\frac{2}{3}\right)^{5-1}$$

$$\frac{64}{27} = a_1 \left(\frac{2}{3}\right)^4$$

$$\frac{64}{27} = a_1 \left(\frac{16}{81}\right)$$

$$\frac{64}{27} \cdot \frac{81}{16} = a_1$$

$$a_1 = 12$$

So, the *n*th term is

$$a_n = a_1 r^{n-1}$$

$$a_n = 12\left(\frac{2}{3}\right)^{n-1}$$

The Sum of a Finite Geometric Sequence

Definition of the Sum of a Finite Geometric Sequence

The sum of a finite geometric sequence with n terms, where the common ratio $r \neq 1$ is

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

Example: Evaluate $\sum_{n=1}^{10} 16\left(\frac{1}{2}\right)^n$.

We can tell that $r = \frac{1}{2}$ because it is the number raised to the power of n. We can also figure out the first term by

$$a_1 = 16\left(\frac{1}{2}\right)^1 = 8$$

Now we can use the formula:

$$S_{10} = 8\left(\frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}}\right) = 8\left(\frac{1-\frac{1}{1024}}{\frac{1}{2}}\right) = 8\left(\frac{.999023}{.5}\right) \approx 15.98$$

Example: Evaluate $\sum_{n=1}^{20} 2(0.1)^n$.

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

$$S_{20} = (0.2)\left(\frac{1-(0.1)^{20}}{1-(0.1)}\right)$$

$$S_{20} = \frac{2}{9}$$

*Note: Your calculator gives you the answer of 0.22222… for the above problem. Use the [MATH] [▶Frac] feature on your calculator to change it to the fraction.

Example: Evaluate $\sum_{n=0}^{12} 4(0.3)^n$

Notice that the index begins at 0 instead of 1. Our formula needs the index to start at 1, so calculate the first term and adjust your formula.

$$\sum_{n=0}^{12} 4(0.3)^n = 4(0.3)^0 + \sum_{n=1}^{12} 4(0.3)^n$$

Now we can use the formula:

$$\begin{aligned}
\sum_{n=0}^{12} 4(0.3)^n &= 4(0.3)^0 + \sum_{n=1}^{12} 4(0.3)^n \\
&= 4 + a_1\left(\frac{1-r^n}{1-r}\right) \\
&= 4 + (1.2)\left(\frac{1-(0.3)^{12}}{1-0.3}\right) \\
&\approx 4 + 1.714 \\
&\approx 5.1714
\end{aligned}$$

<u>Geometric Series</u>

Definition: The summation of the terms of an infinite geometric sequence is called an <u>infinite geometric series</u> or simply a <u>geometric series</u>.

Look at the 2 geometric sequences:

$$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, 8\left(\frac{1}{2}\right)^n, \ldots \qquad r = \frac{1}{2}$$

$$2, 6, 18, 36, 72, 144, 288, \ldots, 2(3^n), \ldots \qquad r = 3$$

The terms of the first sequence are getting smaller and smaller, while the terms of the 2nd sequence are getting larger and larger.

In general, if $|r| < 1$, then the terms approach 0, as $r \rightarrow \infty$.
if $|r| > 1$, then the terms approach ∞, as $r \rightarrow \infty$.

Now look at the 2 infinite series:

$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots + 8\left(\frac{1}{2}\right)^n + \ldots \qquad r = \frac{1}{2}$$

$$2 + 6 + 18 + 36 + 72 + 144 + 288 + \ldots + 2(3^n) + \ldots \qquad r = 3$$

The sum of the 2nd series does not exist because the terms keep getting larger. The sum of the first series does exist, because the terms will eventually get so small that they do not significantly affect the sum.

Look at the general formula for the sum of a finite geometric series.

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ so we get

$$S = a_1\left(\frac{1-r^n}{1-r}\right) = a_1\left(\frac{1-0}{1-r}\right) = a_1\left(\frac{1}{1-r}\right) = \frac{a_1}{1-r}$$

The Sum of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + \ldots$$

has the sum

$$S = \frac{a_1}{1-r}$$

(As noted before, if $|\boldsymbol{r}| > 1$, there is no sum.)

Example: Evaluate $\sum_{n=1}^{\infty} 2(0.4)^{n-1}$.

$$S = \frac{a_1}{1-r} = \frac{2}{1-0.4} = \frac{2}{.6} \approx 3.333\ldots = 3\frac{1}{3}$$

(Use [MATH] [▶Frac] to change to a fraction.)

Example: Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^{n}$.

$$S = \frac{a_1}{1-r} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$$

Example: Find the sum of the series $9+6+4+\frac{8}{3}+\ldots$.

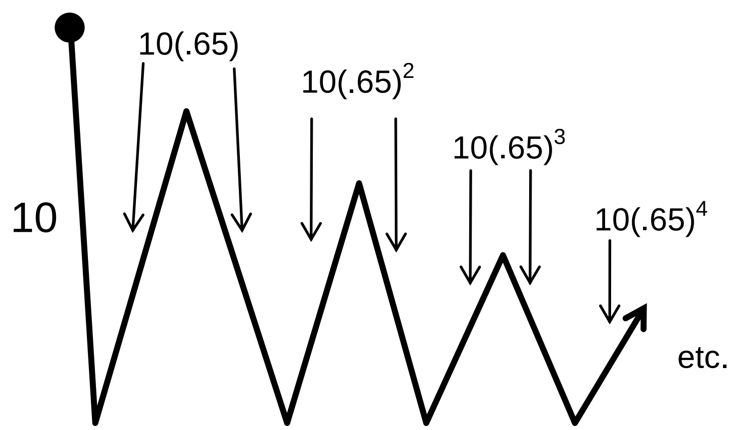
First find *r*: $r = \frac{6}{9} = \frac{2}{3}$

$$S = \frac{a_1}{1-r} = \frac{9}{1-\frac{2}{3}} = \frac{9}{\frac{1}{3}} = 27$$

Applications

Example: A ball is dropped from a height of 10 feet. Each time it bounces back up, it bounces 0.65 times as high as it did on the previous bounce. What is the total distance traveled by the ball?

Solution: Look at the picture:



After the initial drop of 10 feet, we have 2 identical terms for each bounce. We can write the series as

$$10 + 2\sum_{n=1}^{\infty} 10(0.65)^n = 10 + 2\left(\frac{6.5}{1-0.65}\right) \approx 47.14 \text{ feet.}$$