Counting Principles

Example 1: A bag contains six balls, numbered 1 through 6. A ball is drawn from the bag, its number noted, and then it is placed back into the bag. A second ball is drawn and its number noted. In how many different ways can a total of 8 be obtained from the two balls?

Consider the following ordered pairs to be of the form (*first number*, *second number*).

(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)

So, there are five different ways to obtain a total of 8.

Example 2: A bag contains six balls, numbered 1 through 6. A ball is drawn from the bag and its number noted, but it is *not* placed back into the bag. A second ball is drawn and its number noted. In how many different ways can a total of 8 be obtained from the two balls?

Consider the following ordered pairs to be of the form (*first number*, *second number*).

(2, 6), (3, 5), (5, 3), (6, 2)

So, there are four different ways to obtain a total of 8.

What's the difference?

replacement versus *no replacement*

Question: Which offers more choices for license plates?

- a plate with 3 different letters of the alphabet in any order
- a plate with 4 different non-zero digits in any order

Solution: We'll find out at the end of class.

The Fundamental Counting Principle

It is always best to list the possible outcomes, but some events can occur in so many different ways that it is impractical to list them all. In these cases we need to rely on formulas. Our first, and most important, is the Fundamental Counting Principle.

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

*Note that this principle can be extended to any finite number of events.

Example: How many three-letter triplets can be formed from the English alphabet?

26	26	26

$$26 \cdot \bullet\ 26 \bullet 26 = 17{,}576$$

(<u>Note</u>: It is often helpful to draw boxes to represent each event. In the box you enter how many ways the event can occur.)

Example: A diner offers breakfast combination plates which can be made from a choice of one of 4 different types of breakfast meats, one of 8 different styles of eggs, and one of 5 different types of breakfast breads. How many different breakfast combination plates are possible?

4	8	5

$$4 \bullet 8 \bullet 5 = 160$$

Example: How many possible telephone numbers are there in the 952 area code?

8	10	10	-	10	10	10	10

(Phone numbers cannot start with a 0 or a 1.)

$$8 \bullet 10 \bullet 10 \bullet 10 \bullet 10 \bullet 10 \bullet 10 = 8 \bullet 10^6 = 8{,}000{,}000$$

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a permutation of the elements.

Definition of Permutation

A permutation of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example: How many permutations of the letters A, B, C, and D are possible?

4	3	2	1

$$4 \bullet 3 \bullet 2 \bullet 1 = 4! = 24$$

Example: In how many ways 5 singers stand in a row on stage?

5	4	3	2	1

$5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 5! = 120$ ways

Example: How many different baseball hitting line-ups are possible with 9 players?

9	8	7	6	5	4	3	2	1

$9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 9! = 362{,}880$ ways

Number of Permutations of n Elements

The number of permutations of n elements is

$$n \cdot (n-1) \cdots 4 \cdot 3 \cdot 2 \cdot 1 = n!$$

In other words, there are $n!$ different ways that n elements can be ordered.

Sometimes we want to order only part of a set instead of the entire set.

Example: How many permutations are possible from just three of the letters A, B, C, D, and E?

5	4	3

$$5 \bullet 4 \bullet 3 = 60$$

When we want to take r elements out of the collection of n elements and order them, such an ordering is called a <u>permutation of n elements taken r at a time</u>.

**In the above example, we wanted the *number of permutations of 5 elements taken 3 at a time.*

Example: In how many ways can a chairperson, a vice chairperson, and a recording secretary be chosen from a committee of 14 people?

14	13	12

$$14 \bullet 13 \bullet 12 = 2184 \text{ ways}$$

(the number of permutations of 14 people taken 3 at a time)

Permutations of n Elements Taken r at a Time

The number of permutations of n elements taken r at a time is

$$
\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!} \\
&= n(n-1)(n-2)(n-3)\cdots(n-r+1)
\end{aligned}
$$

Example: Evaluate the following.

a) ${}_5P_3$

<u>*solution*</u>: ${}_5P_3 = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = 5\cdot 4\cdot 3 = 60$

b) ${}_{14}P_3$

<u>*solution*</u>: ${}_{14}P_3 = \dfrac{14!}{(14-3)!} = \dfrac{14!}{11!} = 14\cdot 13\cdot 12 = 2184$

**These are the same answers as in the examples above.

Example: Evaluate the following.

a) ${}_7P_5$

<u>*solution*</u>: ${}_7P_5 = \dfrac{7!}{(7-5)!} = \dfrac{7!}{2!} = 7\cdot 6\cdot 5\cdot 4\cdot 3 = 2520$

b) ${}_6P_6$

<u>*solution*</u>: ${}_6P_6 = \dfrac{6!}{(6-6)!} = \dfrac{6!}{0!} = \dfrac{6!}{1} = 6! = 720$

*<u>Note</u>: This is the same as the number of permutations on 6 elements, since we are taking all 6.

<u>Finding Permutations of ***n*** Elements taken ***r*** at a time on a Graphing Calculator</u>

To find ${}_nP_r$ on your calculator, do the following:

1. Type in the value of n on your main screen.
2. Press [MATH] [PRB] [nPr]
3. Type in the value of r on your main screen.
4. Press [ENTER].

Example: Evaluate the previous permutation examples using your calculator.

Example: A bag contains six balls, numbered 1 through 6. A ball is drawn from the bag and its number noted, but it is *not* placed back into the bag. A second ball is drawn and its number noted. In how many different ways can 2 balls be drawn from the bag?

solution: $_6P_2 = \dfrac{6!}{(6-2)!} = \dfrac{6!}{4!} = 6 \cdot 5 = 30$

Example: There are 5 men on the 4x100 relay team (one being the alternate). How many ways can they arrange a 4-man line-up from these 5 men?

solution: $_5P_4 = \dfrac{5!}{(5-4)!} = \dfrac{5!}{1!} = 5! = 120$

Distinguishable Permutations

Example: In how many distinguishable ways can the letters in COMMITTEE be written?

9	8	7	6	5	4	3	2	1

9! = 362,880 ways that the letters can be written

Look at the following 2 possibilities:

$$\text{COM}_1\text{M}_2\text{ITTEE and COM}_2\text{M}_1\text{ITTEE}$$

These 2 “words” are really indistinguishable, but they are both counted in our total of 362,880. So, we need to take our answer of 362,880 and divide it by 2, to remove the duplicates that the 2 M’s cause.

Similarly, we need to divide by 2 to remove duplicates for C and again by 2 to remove duplicates for E.

$$\frac{362{,}880}{2 \cdot 2 \cdot 2} = 45{,}360 \text{ ways}$$

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on with

$$n = n_1 + n_2 + n_3 + \cdots n_k$$

Then the number of distinguishable permutations of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!}$$

In the above example, we had 9 letters total, but we had

two M's, two T's, two E's, one I, one C, and one O

By the formula, then, the number of distinguishable permutations is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!} = \frac{9!}{2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} = \frac{9!}{8} = 45{,}360$$

Example: In how many distinguishable ways can the letters in BANANA be written?

Solution: There are three A's, two N's and one B. So,

$$\frac{6!}{3!\cdot 2!\cdot 1!} = \frac{6!}{12} = 60 \text{ ways}$$

Combinations

Example: From a club of 40 people, how many ways can a president, treasurer, and then a secretary be selected?

solution: This is the permutation ${}_{40}P_3 = \frac{40!}{37!} = 59,280$

or you can use the boxes:

40	39	38

40 • 39 • 38 = 59,280

Example: From a club of 40 people, how many ways can a 3-person committee can be selected?

solution: We start with the permutation

$$ {}_{40}P_3 = \frac{40!}{(40-3)!} = \frac{40!}{37!} = 59{,}280 \text{ ways} $$

Let's say we have selected Bob, Sue, and Tim. These 3 people may have been chosen in any of the following orders:

(Bob, Sue, Tim)	(Bob, Tim, Sue)	(Sue, Tim, Bob)
(Sue, Bob, Tim)	(Tim, Sue, Bob)	(Tim, Bob, Sue)

There are 3! = 6 ways that these 3 people can be chosen.

For this problem, it doesn't matter what order they are chosen in, so we want to consider these all to be the same. That means we need to take our permutation above, and divide it by 3! That gives us the following formula:

$$ \frac{{}_{40}P_3}{3!} = \frac{40!}{3! \cdot (40-3)!} = 9880 \text{ ways} $$

In general, when we want to select a subset of a larger set in which order is not important, we call the subset a combination of n elements taken r at a time.

Combination of n Elements Taken r at a Time The number of combinations of n elements taken r at a time is $${}_nC_r = \frac{n!}{(n-r)!r!}$$

**Note: This is the same formula that we used to find binomial coefficients!

Example: In how many ways can a research team of 4 students be chosen from a class of 14 students?

solution: $${}_{14}C_4 = \frac{14!}{(14-4)!4!} = \frac{14!}{10!4!} = \frac{14\cdot 13\cdot 12\cdot 11}{4\cdot 3\cdot 2\cdot 1} = 1001$$

Example: You are told that on a given test you must answer 10 out of the 12 questions. How many groups of 10 questions are possible?

<u>*solution*</u>: $_{12}C_{10} = \dfrac{12!}{(12-10)!10!} = \dfrac{12!}{2!10!} = \dfrac{12 \cdot 11}{2 \cdot 1} = 66$

Example: A building company is hiring extra summer help. They need 4 additional employees to work outside in the lumber yard and 3 to work inside the store. In how many ways can these positions be filled if there are 10 applications for the outside work and 5 for the inside work?

This problem combines the basic counting principle and also combinations.

There are $_{10}C_4 = 210$ ways to fill the outside work.
There are $_5C_3 = 10$ ways to fill the inside work.

Then by the basic counting principle, there are

210 • 10 = 2100 ways to fill both the inside and outside positions.

Example: Find the total number of subsets of a set that has 5 elements.

Solution: We need to consider the number of subsets with 0 elements, 1 element, 2 elements, etc. all the way up to subsets that have 5 elements.

Number of subsets with 0 elements = ${}_5C_0 = 1$.
Number of subsets with 1 elements = ${}_5C_1 = 5$.
Number of subsets with 2 elements = ${}_5C_2 = 10$.
Number of subsets with 3 elements = ${}_5C_3 = 10$.
Number of subsets with 4 elements = ${}_5C_4 = 5$.
Number of subsets with 5 elements = ${}_5C_5 = 1$.

The total number of subsets = 32.

***In general, the total number of subsets of a set of n elements is 2^n.

Example: You are dealt 5 cards from an ordinary deck of 52 playing cards. In how many ways can you get a full house? (3 of one kind and 2 of another)

The # of ways to select 3 of 4 cards for 3 of a kind: ${}_4C_3 = 4$
The # of ways to select the type of card: ${}_{13}C_1 = 13$
The # of ways to select 2 of 4 cards for 2 of a kind: ${}_4C_2 = 6$
The # of ways to select the type of card: ${}_{12}C_1 = 12$

4 • 13 • 6 • 12 = 3744 ways to get a full house

Tips for Counting

When solving problems involving counting principles, as the following questions:

1. Is the order important? (yes = *permutation*)

2. Are the chosen elements a subset of a larger set in which order is not important? (yes = *combination*)

3. Does the problem involve two or more separate events? (yes = *Fundamental Counting Principle*)